Math 148 Notes

Eason Li

2024 W

Lecture 2 - Wed - Jan 10 - 2024

Section 1

Key to uniqueness of solutions below

Rolle's theorem which implies if

$$\frac{dG}{dx} = 0$$

then G is locally constant, i.e. if we focus on a point, the graph near that point is horizontal. You should know the geometric idea behind the proof.

Section 2

General solutions to

$$\frac{dF}{dx} = f$$

are given by a special solution plus a locally constant function. The set of solutions is called the indefinite integral of f and denoted as

$$\int f(x)dx$$

Section 3

The solution to

$$\frac{dF}{dx} = f, F(a) = 0 \text{ over } [a, b]$$

is given by $F_0(x) - F_0(a)$ where F_0 is any choice of antiderivative. It is usually called the definite integral and denoted as

$$\int_a^x f(t)dt = F_0(x) - F_0(a)$$

Example 0.1

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

Hint: thinking about the unit circle.

Section 4

Proposition 0.1

If $f(x)$ is an even function over $(-\infty,\infty)$, i.e. $f(-x)=f(x)$, then $F(x)=\displaystyle\int_0^x f(t)\,dt$ is odd.

Proof:
Goal: $G(x)=F(-x)+F(x)=0$
Note: $\frac{d}{dx}G(x)=0 \;\Rightarrow\; G(x)$ is constant function, and $G(0)=0$. □

Proposition 0.2

Exercise: If $f(x)$ is an odd function over $(-\infty,\infty)$, then $F(x)=\displaystyle\int_0^x f(t)\,dt$ is even.

Lecture 3 - Fri - Jan 12 - 2024

We introduced some basic integration in the class, but I am a little too lazy to copy them down since they can be found online easily.

Exercise: Prove:

$$\frac{d}{dx}\arctan(x) = \frac{1}{\sqrt{1-x^2}}$$

Proof: We let $y=\arcsin(x)$ and hence $x=\sin(y)$, therefore, we obtain

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \overset{①}{=} \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

Remark: $\frac{dx}{dx}=\frac{d\sin(y)}{dx} \;\Rightarrow\; 1=y'\cos(y) \;\Rightarrow\; y'=\frac{1}{\cos(y)}$.

as desired. □

Example 0.2

$$\begin{aligned}\int \frac{1}{(x-1)(x-2)}\,dx &= \int \frac{1}{x-2}-\frac{1}{x-1}\,dx \\ &= \ln|x-2|+\ln|x-1|+C\end{aligned}$$

Example 0.3

$$\begin{aligned}\int \frac{1}{\sin^2(x)\cos^2(x)}\,dx &= \int \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)\cos^2(x)} \\ &= \tan(x) - \cot(x) + C\end{aligned}$$

However, sometimes breaking things up, so we need to introduce other tricks to make our life better.

Method 1: Substitution a.

Definition 0.1

$$\frac{d\,F\circ G}{dx} = (F'\circ G)\left(\frac{dG}{dF}\right) \quad\Rightarrow\quad \int F'\circ G\,\frac{dG}{dF} = F\circ G + C$$

Example 0.4

$$\begin{aligned}\int \frac{e^x}{1+e^x}\,dx &= \frac{1}{1+e^x}\,d\left(1+e^x\right) \\ &= \ln\left|1+e^x\right| + C\end{aligned}$$

Method 1: Substitution b.

Definition 0.2

$$\int f\circ\varphi(t)\varphi'(t)\,dt = G(t) + C \quad\Rightarrow\quad \int f(x)\,dx = G\circ\varphi^{-1}(x) + C$$

Proof:

$$\begin{aligned}\frac{d\,G\circ\varphi^{-1}(x)}{dx} &= \frac{dG}{dt}\cdot\frac{d\varphi^{-1}}{dt} \\ &= f\circ\varphi\cdot\frac{d\varphi}{dt}\cdot\frac{d\varphi^{-1}}{dx} = f\circ\varphi\end{aligned}$$

as desired. $\square$

Example 0.5

Evaluate

$$\int \frac{1}{\sqrt{e^x+1}}\,dx$$

We replace $t = \sqrt{e^x + 1} \Rightarrow x = \ln(t^2 - 1)$. Therefore the original expression is equal to

$$\begin{aligned}
\int \frac{1}{\sqrt{e^x + 1}}\, dx &= \int \frac{1}{t}\, d\left(\ln(t^2 - 1)\right) \\
&= \int \frac{2}{t^2 - 1}\, dt \\
&= \int \frac{(t+1) - (t-1)}{(t+1)(t-1)}\, dt \\
&= \ln\left|\frac{t-1}{t+1}\right| + C
\end{aligned}$$

Method 2

Definition 0.3

We note that

$$\begin{aligned}
& \frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx} \\
\Rightarrow \quad & fg + C = \int \frac{df}{dx} g\, dx + \int f \frac{dg}{dx}\, dx \\
\Rightarrow \quad & fg + C = \int g\, df + \int f\, dg \\
\Rightarrow \quad & \int f\, dg = fg - \int g\, df
\end{aligned}$$

Example 0.6

$$\begin{aligned}
\int e^x \sin(x)\, dx &= \int \sin(x)\, d\left(e^x\right) \\
&= e^x \sin(x) - \int e^x\, d(\sin(x)) \\
&= e^x \sin(x) - \int e^x \cos(x)\, dx \\
&= e^x \sin(x) - \int \cos(x)\, d\left(e^x\right) \\
&= e^x \sin(x) - e^x \cos(x) - \int e^x\, d(\cos(x)) \\
&= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)\, dx
\end{aligned}$$

Therefore

$$\int e^x \sin(x)\, dx = \frac{e^x}{2}\left(\sin(x) - \cos(x)\right)$$

Lecture 4 - Mon - Jan 15 - 2024

Exercise: Show that

$$\frac{\mathrm{d}\left(\mathrm{f}^{-1}(\mathrm{y})\right)}{\mathrm{dy}} = \frac{1}{f' \circ f^{-1}(y)}$$

Proof: We know that

$$\begin{aligned}
\left(\frac{d\,f^{-1}}{d\,y}\right)\left(\frac{d\,y}{d\,f^{-1}}\right) &= 1 \\
\Rightarrow \quad \frac{d\,f^{-1}}{d\,y}\left(\frac{d\,f \circ f^{-1}}{d\,f^{-1}}\right) &= 1 \\
\frac{d\,f^{-1}}{d\,y}\left(f'\left(f^{-1}(y)\right)\right) &= 1 \\
\Rightarrow \quad \frac{d\,f^{-1}}{d\,y} &= \frac{1}{f' \circ f^{-1}(y)}
\end{aligned}$$

as desried. □

Exercise: Show that

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - F\left(f^{-1}(x)\right)$$

Proof: We know that

$$\begin{aligned}
\int f^{-1}(x)\,dx &= xf^{-1}(x) - \int xd\left(f^{-1}(x)\right) \\
&= xf^{-1}(x) - \int f\left(f^{-1}(x)\right) d\left(f^{-1}(x)\right) \\
&= xf^{-1}(x) - F\left(f^{-1}(x)\right)
\end{aligned}$$

as desired. □

Example 0.7

Evaluate

$$\int \frac{xe^x}{(1+x)^2}\,dx$$

Proof: There are two ways of doing this problem:

1. Method 1:

$$\begin{aligned}
\int \frac{xe^x}{(1+x)^2}\,dx &= -\int xe^x\,d\left(\frac{1}{1+x}\right) \\
&= -\frac{xe^x}{1+x} + \int \frac{1}{1+x}\,d\left(xe^x\right) \\
&= -\frac{xe^x}{1+x} + \int \frac{xe^x + e^x}{1+x}\,dx \\
&= -\frac{xe^x}{1+x} + e^x + C \\
&= \frac{e^x}{1+x}
\end{aligned}$$

2. Method 2:

   Notice that

$$\begin{aligned}\int \frac{e^x}{1+x}\,dx &= \frac{1}{1+x}\,d\left(e^x\right)\\ &= \frac{e^x}{1+x} + \int \frac{e^x}{(1+x)^2}\,dx\end{aligned}$$

   Therefore, we can obtain that

$$\begin{aligned}\int \frac{xe^x}{(1+x)^2}\,dx &= \int \frac{e^x}{1+x}\,dx - \int \frac{e^x}{(1+x)^2}\,dx\\ &= \frac{e^x}{1+x}\end{aligned}$$

as desired. □

Problem 1. $\dfrac{d}{dx}\arcsin x$

Proof: Fundamental. □

Problem 2. $\displaystyle\int (e^x+2)^{10}\ dx$

Proof: Let $t = e^x + 2$, so

$$\begin{aligned}\int (e^x+2)^{10}\ dx &= \int t^{10}\ d\left(\ln(t-2)\right)\\ &= \int \frac{t^{10}}{t-2}\ dt\end{aligned}$$

□

Problem 3. $\displaystyle\int \frac{1}{1-x^2}\ dx$

Proof:

$$\int \frac{1}{1-x^2}\ dx = -\int \frac{1}{x^2-1}\ dx = -\frac{1}{2}\int \frac{(x+1)-(x-1)}{(x+1)(x-1)}\ dx$$

boom, one liner. □

Problem 4. Find $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n)!}$ **given** $\displaystyle\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

Proof: Solution $= -x\sin x + \cos x$. □

Problem 5. $\displaystyle\int \ln x\ dx$

Proof:

$$\begin{aligned}\int \ln x\ dx &= x\ln x - \int x\ d\left(\ln x\right)\\ &= x\ln x - \int\ dx\\ &= x\ln x - x + C\end{aligned}$$

too trivial. □

Problem 6. $\int e^x \cos x \, dx$

Proof: Check Lecture 3 example 0.6 □

Lecture 5 - Wed - Jan 17 - 2024

Hardly any lecture notes today. 💀

One of the most important example would be

Example 0.8

We would like to figure out what the following is:

$$\int \frac{1}{(t^2+1)^k} \, dt$$

Here, we need to use a method that brings down the power "k", and thus we can evaluate the indefinite integral using **"induction"**. (However, due to the low time efficiency of this method, only use it when you have no other choices).

Proof: We evaluate

$$\begin{aligned}
\int \frac{1}{(t^2+1)^k} \, dt &= \int \frac{t^2+1-t^2}{(t^2+1)^k} \, dt \\
&= \int \frac{1}{(t^2+1)^{k-1}} \, dt - \int \frac{t^2}{(t^2+1)^k} \, dt \\
&= \int \frac{1}{(t^2+1)^{k-1}} \, dt - \frac{1}{k-1} \int t^2 \cdot \frac{1}{2t} \, d\left(\frac{1}{(t^2+1)^{k-1}}\right) \\
&= \int \frac{1}{(t^2+1)^{k-1}} \, dt - \frac{1}{2k-2} \left(\frac{t}{(t^2+1)^{k-1}} - \underbrace{\int \frac{1}{(t^2+1)^{k-1}} \, dt}_{\bigstar} \right)
\end{aligned}$$

Notice that we can evaluate the leftmost part easily, and we have successfully brought the power down to $k-1$. Thus, repeating the process will eventually bring us the desired solution (Just tedious :3). □

Example: We will talk about how we can change the variable using trigonometry next lecture.

Lecture 6 - Fri - Jan 19 - 2024

Recall from last lecture, , we try to solve $\int \frac{1}{(1+x^2)^2} \, dx$, the trick is that the above expression is equal to $\int \frac{1+x^2-x^2}{(1+x^2)^2} \, dx$, and thus we can break the question down and bring the exponent down by 1.

Exercise: Do the same trick for $\int \frac{1}{(1+x^2)^3} \, dx$.

In general, if you have a rational function, $\frac{Q(x)}{P(x)}$ such that $\deg(Q) < \deg(P)$, we can always break the question down by factor the denominator and group the like terms (this is the algorithm behind how computer solve such problems).

However, there are also other cases, suppose we have $\int R(\cos(\theta), \sin(\theta))\, d\theta$,

Example: $R(x,y) = \frac{x+y}{x^3+y^3}, \frac{1}{x+y}, \frac{x^7}{x^5+6y+8y^6}, \dots$

Weierstrass substitution

There is a trick called **Weierstrass substitution**, by taking $t = \tan\frac{\theta}{2}$, we can simplify the question a lot.

Remark: The intuition behind this method is the unit circle.

Therefore, we want to solve the system of equations

$$\begin{cases} y = t(x+1) \\ x^2+y^2 = 1 \end{cases} \Rightarrow \begin{cases} x^2 + t^2(x+1)^2 = 1 \\ (1+t^2)x^2 + 2t^2x + t^1 - 1 = 0 \end{cases}$$

Since we know that one of the solution is $x_1 = -1$, we can then obtain that (by Vieta's Theorem) the other solution is $x_2 = \frac{1-t^2}{1+t^2}, y_2 = \frac{2t}{1+t^2}$. And this implies that

$$(\cos(\theta), \sin(\theta)) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

Therefore, we can substitute the solution back into the integral and obtain $\int R(\cos(\theta), \sin(\theta))\, d\theta = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\, d\theta$, and after solving for $d\theta$ (recall that we have $t = \tan\frac{\theta}{2}$), we simplify the question to just solving for rational functions (which we are more familiar with):

$$\int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \frac{2\, dt}{1+t^2}$$

Example 0.9

We can also derive them in the direct way to check out solutions:

$$\begin{aligned} x = \cos(\theta) &= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} \\ &= \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\ &= \frac{1-t^2}{1+t^2} \end{aligned} \qquad \begin{aligned} y = \sin(\theta) &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} \\ &= \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \\ &= \frac{2t}{1+t^2} \end{aligned}$$

Example: Suppose we have $R(x, \sqrt{1-x^2})\, dx$, we let $x = \cos\theta$, then we can obtain that

$$R(x, \sqrt{1-x^2})\, dx = -R(\cos\theta, \sin\theta)\sin\theta\, d\theta$$

Example: Suppose we have $R(x, \sqrt{x^2-1})\,dx$, we let $x = \frac{e^u+e^{-u}}{2}$, then we can obtain that

$$\begin{aligned} R(x, \sqrt{x^2-1})\,dx &= R\left(\frac{e^u+e^{-u}}{2}, \frac{e^u-e^{-u}}{2}\right)\frac{e^u-e^{-u}}{2}\,du \\ &= R\left(\frac{e^u+e^{-u}}{2}, \frac{e^u-e^{-u}}{2}\right)\frac{1-(e^{-u})^2}{2}\,d\left(e^u\right) \end{aligned}$$

And then we let $t = e^t$, we would then obtain a rational function that we can easily integrate.

Feymann's Trick

It is important for us to remind you that this is the trick a physicist used 💀.

Example 0.10

Suppose we want to find $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$. The trick is that assume we want to compute something like $F(t) = \displaystyle\int_0^\infty \frac{\sin x}{x}e^{-tx}\,dx$, so our goal is to find $F(0)$. Taking the derivative of it to obtain $F'(t) = -\displaystyle\int_0^\infty \sin x e^{-tx}\,dx$, which yields us the answer: $\displaystyle\frac{t\sin x+\cos x}{1+t^2}e^{-tx}\Big|_0^\infty = -\frac{1}{1+t^2}$. Moreover, we know that $F(\infty) = 0$, so we can get $F(t) = \arctan t + \frac{\pi}{2}$, which gives us that $F(0) = \frac{\pi}{2}$.

Tut 1.1 - Fri - Jan 19 - 2024

I dont guarantee everything in this section is correct, I just copied :3
Suno's (Ethan's) version of notes is called `riemann_integrable_countable_disc.pdf`

When is a function integrable?

We first need to know some definitions:

Definition 0.4: Countable

S is **countable** if there is a bijection between $\mathbb{N}$ and S.
Example: $\mathbb{Q}$ is countable, while $\mathbb{R}$ is not countable.

Definition 0.5: Open & Closed

For an **open interval** U, if $x \in U$, then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq U$.
We call an interval C **closed** if $\mathbb{R}\backslash C$ is open.
Example: (0, 1) is open, while [0, 1] is closed.

Definition 0.6: Oscillation

The **oscillation** on an open interval U is

$$\omega_f(U) = \sup_{y,z\in U} \big|f(y) - f(z)\big|$$

and the **oscillation** at a point x_0 is

$$\omega_f(x_0) = \lim_{h\to 0} \omega_f(x_0 - h, x_0 + h)$$

Remark: **Oscillation** at a point is a measure of how discontinuous the function is.

Result 0.1

What does it maean to have $\omega_f(x_0) = 0$, this means that we would have $\lim\limits_{h\to 0} \omega_f(x_0 - h, x_0 + h) = 0$, which is equivalent to

$$\lim_{h\to 0} \sup_{y,z\in(x_0-h,x_0+h)} \big|f(y) - f(z)\big| = 0$$

hence if h is δ, then we would have $\big|f(y) - f(z)\big| < \epsilon$, which means the function is continuous!

Example 0.11: Removable discontinuity

Suppose we have function

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 6 & x = 2 \end{cases}$$

Therefore, we have $\omega_f(2) = 6 - 4 = 2$.

Example 0.12: Jump discontinuity

Suppose we have function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

Therefore, we have $\omega_f(0) = 2$.

Example 0.13: Essential discontinuity

Suppose we have function

$$f(x) = \frac{1}{x}$$

therefore we have $\omega_f(0) = \infty$.

Definition 0.7: Compactness

A set K is compact if for every **open cover** on K there exists a **finite subcover**. We mean open cover by

$$\Big\{U_i\Big\}_{i=1}^{\infty} \quad s.t. \quad K \subseteq \bigcup_{i=1}^{\infty} U_i$$

Definition 0.8: Reachable

We say that $x \in [a,b]$ is reachable if $[a,x]$ can be covered by finitely manu open sets

Theorem 0.1: Heine-Burel

If a set S is closed and bounded, then S is compact.

Proof: To prove the theorem, we have two steps

1. Prove that $[a,b]$ is compact
2. Prove that $S \subseteq [a,b]$ is compact

To prove (1), if b is reachable, then we are done. If b is not reachable, then let $x \in \Big\{x \in [a,b] : x \text{ is not reachable}\Big\}$, thus there exists a greatest lower bound x_0. We take some small interval $[a_1, b_1] \ni x_0$. By archimedian property there exist x_1, x_2 such that $a_1 < x_1 < x_0 < x_2 < b_1$. Thus we have

$$\begin{aligned} x_2 > x_0 \quad &\to \quad x_2 \text{ is not reachable} \\ x_1 < x_0 \quad &\to \quad x_1 \text{ is reachable so } [a, x_1] \text{ can be finitely covered} \end{aligned}$$

Take covering for $[a, x_1]$ and add the open set (a_1, b_1), so $[a, x_2]$ is finitely covered, which implies that x_2 is reachable, contradicting our assumption. Therefore $[a,b]$ is compact.

Now we want to prove (2). If S closed, and $S \subseteq [a,b]$, we let $\Big\{U_i\Big\}_{i=1}^{\infty}$ be open cover for S, then $\Big\{U_i\Big\}_{i=1}^{\infty} \cup \Big\{[a,b] \backslash S\Big\}$ is an open cover for $[a,b]$. Then there exists finite subcover of $[a,b]$ given by $U_i, \ldots, U_{iN}, [a,b]\backslash S$, so S is covered by $U_i, \ldots, U_{iN}$. $\square$

Now we can go back to our original question.

Let $f : [a,b] \to \mathbb{R}$ have countably many discontinuities. Define $D_s = \Big\{x \in [a,b] : \omega_f(x) \geq s\Big\}$ for some $s > 0$. Take some $x_0 \in [a,b]\backslash D_s$ we would have $\omega_f(x_0) = t < s$, then

$$\sup_{\substack{y,z \in \\ (x_0 - h, x_0 + h)}} \Big| f(y) - f(z) - t \Big| < \epsilon$$

for small h. For $y \in (x_0 - h, x_0 + h)$, we have

$$\sum_{\cdots} \Big| f(y) - f(z) \Big| < t + \underbrace{\epsilon}_{\frac{s-t}{2}} \leq \frac{s+t}{2}$$

which implies that $\omega_f(y) < s$ for $y \in (x_0 - h, x_0 + h)$, which further implies that $[a,b]\backslash D_s$ is open.

Therefore we have $D_s \subseteq [a,b]$ is closed is bounded, which implies compactness, and it is countable because it is the subset of all discontinuities.

$$D_s = \Big\{S_i\Big\}_{i=1}^{\infty} \qquad S \subseteq \bigcup_{i=1}^{\infty} \underbrace{\left(S_i - \frac{\epsilon}{2^{i+2}(M-m)}, S_i + \frac{\epsilon}{2^{i+2}(M-m)}\right)}_{I_i}$$

where $M = \sup\limits_{x \in [a,b]} f(x)$ and $m = \sup\limits_{x \in [a,b]} f(x)$. We then define

$$\underbrace{[a,b]\backslash \bigcup_{j=1}^{N} I_{ij}}_{\omega_f(x_0)<s}$$

to be C, then there exist δ_{x_0} such that for all $y, z \in (x_0 - \delta_{x_0}, x_0 + \delta_{x_0})$ we have $\Big|f(y) - f(z)\Big| < s$. Therefore

$$\Big\{(x_0 - \delta_{x_0}, x_0 + \delta_{x_0}) : x_0 \in C\Big\}$$

is an open cover for C, and because it is closed and bounded, so it is compact. Hence there exist finite subcover

$$(x_0 - \delta_{x_0}, x_0 + \delta_{x_0}), \ldots, (x_k - \delta_{x_k}, x_k + \delta_{x_k})$$

so

$$\underbrace{x_i - \delta_{x_i}, x_i + \delta_{x_i}}_{C_2} \quad and \quad \underbrace{I_{i_1}, \ldots, I_{i_N}}_{C-2}$$

together cover $[a,b]$. We can then choose a partition P of $[a,b]$ such that each interval

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{C_1} \Big(M(y_i) - m(y_i)\Big)\Delta y_i + \sum_{C_2} \Big(M(y_i) - m(y_i)\Big)\Delta y_i \\ &\leq (M-m)\sum_{C_1} \Delta y_i + (M-m)\sum_{C_2} \Delta y_i \end{aligned}$$

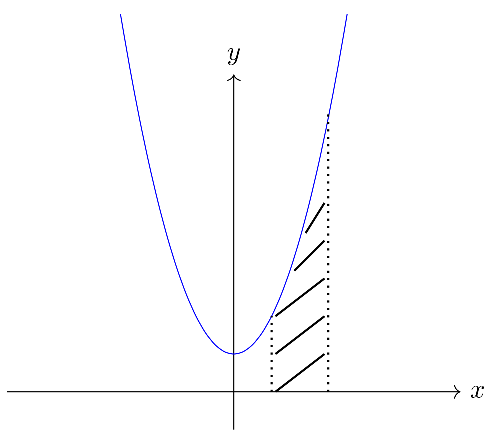
Moreover, we know

$$
\begin{aligned}
(M-m)\sum_{C_1}\Delta y_i &\le (M-m)\sum_{j=1}^{N}\Delta I_{i_j} \\
&\le (M-m)\sum_{j=1}^{\infty}\Delta I_{i_j} \\
&\le (M-m)\sum_{j=1}^{\infty}\frac{\epsilon}{2^i(M-m)} \\
&\le (M-m)\frac{\epsilon}{2(M-m)} \\
&= \frac{\epsilon}{2} \\
(M-m)\sum_{C_2}\Delta y_i &\le s\sum_{C_2}\Delta y_i \\
&\le s\sum_{C_2}(b-a)
\end{aligned}
$$

and because we know that s is arbitrary, so we can simply choose s to be $\dfrac{\epsilon}{2(b-a)}$. ■

Lec 7 - Mon - Jan 22 - 2024

Given function f continuous and the graph as shown below,



Goal:
Define the area of the shaded region

$A(x)$

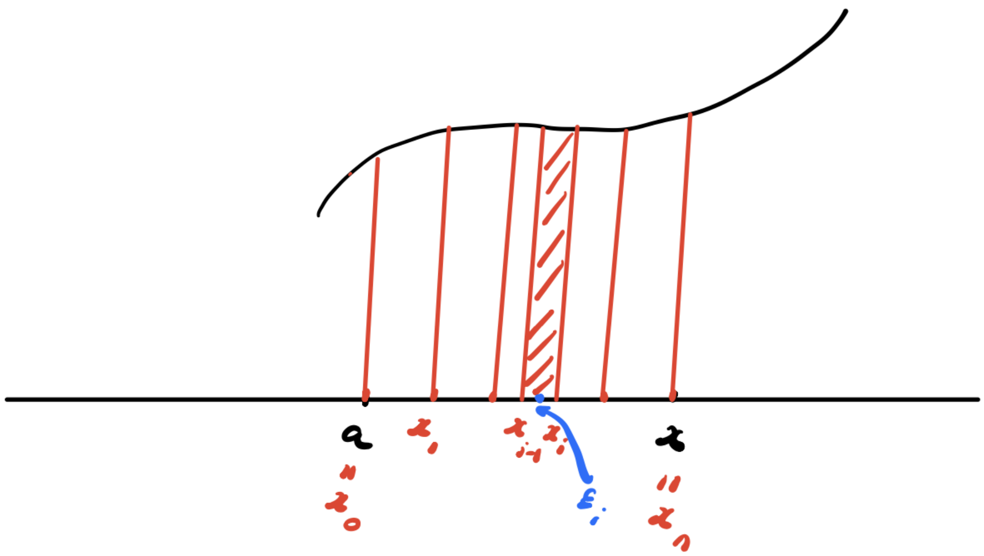
and show that $\frac{d\,A(x)}{dx} = f$

Remark: The intuition lying behind this is that assume $A(x)$ is well-defined, then we would have

$$\begin{aligned}\frac{dA}{dt} &= \lim_{h\to 0}\frac{A(x+h)-A(x)}{h}\\ &= \lim_{h\to 0}\frac{h\cdot f(x')}{h} \quad \text{for } x'\in[x,x+h]\\ &= f(x)\end{aligned}$$

since x' is getting close to x?

Riemann Sum



Suppose we have partition

$$\begin{aligned}T &: x_0 = a < x_1 < \cdots < x_n = x,\\ ||T|| &= \max_{1\le i\le n}\Delta x_i\\ &= \max_{1\le i\le n}\left|x_{i+1}-x_i\right|\end{aligned}$$

then we have

$$S(T,\xi)=\sum_{i=1}^{n} f(\xi_i)\Delta x_i$$

We also define

$$A(x) := \int_a^x f(t)\;dt = \lim_{||T||\to 0} S(T,\xi)$$

and we know that if $\lim_{||T||\to 0} S(T,\xi)$ exists, then there exists $A(x)$ so that for any $\epsilon>0$, there exists $\delta>0$ such that if $||T||<\delta$, then

$$\left|A(x)-S(T,\xi)\right|<\epsilon$$

We know that $\lim_{||T||\to 0} S(T,\xi)$ is "indeed well-defined" (Chen, 2024), and we define

$$\overline{S}(T) = \sum_{1\le i\le n} M_i \Delta x_i \qquad M_i = \sup_{[x_i, x_{i+1}]} f(x)$$
$$\underline{S}(T) = \sum_{1\le i\le n} m_i \Delta x_i \qquad M_i = \inf_{[x_i, x_{i+1}]} f(x)$$

Result 0.2

Therefore we have the following result

$$\underline{S}(T) \le S(T,\xi) \le \overline{S}(T)$$

Example 0.14: Key observation:

Given two partitions T and T', we have

$$\underline{S}(T) \le \underline{S}(T\cup T') \le \overline{S}(T\cup T') \le \overline{S}(T)$$

because of the **monotonocity of the partition.**

Therefore, now we want to show that

$$\lim_{||T||\to 0} \underline{S}(T)$$

exists.

Proof: We need to show that $\lim_{||T||\to 0} \underline{S}(T) = \sup_T \underline{S}(T)$. SFAC that the above statement does not hold, then we would have $\lim_{||T_i||\to 0} \underline{S}(T_i) < \sup_T \underline{S}(T)$. Therefore, for some T', we have $\lim_{||T_i||\to 0} \underline{S}(T_i) < \underline{S}(T')$. Note that $\underline{S}(T') \le \underline{S}(T_i \cup T') \approx \underline{S}(T_i)$ for i large, which implies that $\underline{S}(T') \le \lim_i \underline{S}(T_i)$. $\square$

Remark: Similarly, we can find that $\lim_{||T||\to 0} \overline{S}(T)$ exists too (by taking the negative of the original function and argue for the $\lim_{||T||\to 0} \underline{S}(T)$) again.

Theorem 0.2

$\lim_{||T||\to 0} \overline{S}(T) = \lim_{||T||\to 0} \underline{S}(T)$

Proof: It is equivalent to show that

$$\lim_{||T||\to 0} \sum_i (M_i - m_i)\Delta x_i = 0$$

Since f is continous, then for all $\epsilon > 0$, there exists $\delta > 0$ such that for $z_1 - z_2 < \delta$ we would have $f(z_1) - f(z_2) < \epsilon$. Hence we let the partition to be smaller than δ, and thus we have $M_i - m_i < \epsilon$, which implies

$$\lim_{||T||\to 0} \sum_i (M_i - m_i)\Delta x_i = 0 < \epsilon$$

and thus we have $\lim \sum_i M_i \Delta x_i = 0 \lim \sum_i m_i \Delta x_i = 0$. $\square$

Lec 8 - Wed - Jan 24 - 2024

Chen noticed that the last lecture was not rigorous enough (plus the fact that we are waaaaaay ahead of the other section), so he decided to do the proofs "a lil more" rigorously :3.

Recall from last lecture, we want to prove that

Goal 0.1

Show that $\lim_{||T_i||\to 0} \underline{S}(T) = \sup_T \underline{S}(T)$ exists

Proof: SFAC that the limit does not approach the desired value, instead, suppose we have

$$\lim_{||T_i||\to 0} \underline{S}(T) < \sup_T \underline{S}(T)$$

Hence we can find a new fixed partition T' ("this is true" - Chen 2024) such that we have

$$\lim_{||T_i||\to 0} \underline{S}(T) < S(T') \leq \sup_T \underline{S}(T)$$

which is equivalent to saying that $S(T')$ lies in between $\lim_{||T_i||\to 0} \underline{S}(T)$ and $\sup_T \underline{S}(T)$. Nevertheless, we can also find that the total difference between

$$\lim_{||T_i||\to 0} \underline{S}(T_i) - \lim_{i\to\infty} \underline{S}(T_i \cup T') \leq \lim_{||T_i||\to 0} \left(\underbrace{\max|f|}_{fixed} \cdot ||T_i|| \cdot \underbrace{|T'|}_{fixed} \right)$$
$$= 0$$

Therefore we concluded that $\lim_{||T_i||\to 0} \underline{S}(T)$ is arbitrarily close to $\underline{S}(T')$ for whatever partition T' we choose, and thus we can conclude that $\lim_{||T_i||\to 0} \underline{S}(T) = \sup_T \underline{S}(T)$ exists. $\square$

Exercise: Using the similar argument, show that $\lim_{||T_i||\to 0} \overline{S}(T_i) = \inf_T \overline{S}(T)$

Goal 0.2

Show that $\lim_{||T||\to 0} \overline{S}(T) = \lim_{||T||\to 0} \underline{S}(T)$

Proof: It is equivalent to showing that

$$\lim_{||T||\to 0} \left(\underline{S}(T) - \overline{S}(T) \right) = 0$$

which is also equivalent to showing that

$$\lim_{||T||\to 0}\left(\sum_{i=1}^{n\approx\infty}(M_i - m_i)\Delta x_i\right) = 0$$

To show the above equation, we need to recall that we assume function f to be continous and bounded, thus by *Heine-Cantor theorem*, it is uniformly continuous on a closed bounded interval. Therefore, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that for all $z_1, z_2 \in [a, x]$, if $z_1 - z_2 < \delta$, we would have $f(z_1) - f(z_2) < \epsilon$. This means a lot to us because now for our small enough partition, we have

$$\begin{aligned}\sum_{i=1}^{n}(M_i - m_i)\Delta x_i &\le \epsilon\sum_{i=1}^{n}\Delta x_i \\ &= \epsilon(b-a)\end{aligned}$$

which trivially approaches 0 as ϵ gets indefinitesmally small. Therefore, as a result, we obtain that $\lim\limits_{||T||\to 0}\overline{S}(T) = \lim\limits_{||T||\to 0}\underline{S}(T)$ is indeed true. □

Result 0.3

1. If f is bounded, then $\lim\limits_{||T||\to 0}\overline{S}(T) = \lim\limits_{||T||\to 0}\underline{S}(T)$ are both well-defined.
2. Definition below.

Definition 0.9: Riemann Integrable

Given f bounded over $[a, x]$, if $\lim\limits_{||T||\to 0}\overline{S}(T) = \lim\limits_{||T||\to 0}\underline{S}(T)$,
then f is called **Riemann integrable** over $[a, x]$.

Exercise: Show that an increasing function over $[a, x]$ is Riemann integrable.
Proof:

□

Lec 8.1 (Shum section) - Wed - Jan 24 - 2024

Fundamental Theorem of Calculus

Theorem 0.3: FTC

1. Suppose F is differentiable and riemann integrable, then we have

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

2. Suppose f is continuous and integrable, then we have

$$\frac{d}{dx}\int_a^x f(t)\ dt\Big|_{x_0} = f(x_0)$$

Proof:

$\square$

Theorem 0.4: Extended FTC2

Suppose function f is continuous on $[a, b]$ and $g : [c, d] \to [a, b]$ be differentiable on $[c, d]$, then

$$\frac{d}{dx}\int_a^{g(x)} f(t)\ dt = f(g(x))g'(x) \qquad \forall\ x \in (c, d)$$

Proof: Define $G(y) = \displaystyle\int_a^y f(t)\ dt$, and thus $F(x) = \displaystyle\int_a^{g(x)} f(t)\ dt = (G \circ g)(x)$. Therefore, by chain rule, we can obtain that

$$F'(x) = G'(g(x))g'(x)$$

as desired. $\square$

Example 0.15

Differentiate the sine integral function $Si(x) = \displaystyle\int_a^x \frac{\sin(t)}{t}\ dt$.

Proof: We should interpret this as the integral of $f(x) = \begin{cases} \frac{1}{x}\sin(x) & x \neq 0 \\ 1 & x = 0 \end{cases}$. By FTC2, we then have

$$\frac{d}{dx}Si(x)\Big|_{x_0} = f(x_0) = \begin{cases} \frac{1}{x_0}\sin(x_0) & x_0 \neq 0 \\ 1 & x_0 = 0 \end{cases}$$

as desired. $\square$

Example 0.16

Evaluate $\displaystyle\frac{d}{dx}\int_1^x \frac{\sin(t)}{t}\ dt$

Proof: We simply have

$$\int_1^x \frac{\sin(t)}{t}\ dt = \int_0^x \frac{\sin(t)}{t}\ dt - \underbrace{\int_0^1 \frac{\sin(t)}{t}\ dt}_{constant\ term}$$

as desired $\square$

Example 0.17

Evaluate $\frac{d}{dx}\int_0^{x^2}\frac{\sin(t)}{t}\,dt$

Proof: We have

$$\begin{aligned}\frac{d}{dx}\int_0^{g(x)}\frac{\sin(t)}{t}\,dt &= \frac{\sin(g(x))}{g(x)}\cdot g'(x)\\ &= \frac{\sin(x^2)}{x^2}\cdot 2x\\ &= \frac{2\sin(x^2)}{x} \qquad x\neq 0\end{aligned}$$

However, when $x=0$. we have $g(x)=0^2=0$, $g'(x)=2(0)=0$, and $f(g(0))=1$, thus

$$\frac{d}{dx}\int_0^{x^2}\frac{\sin(t)}{t}\,dt\bigg|_{x=0}=(1)(0)=0$$

as desired. □

Lec 9 - Fri - Jan 26 - 2024

Theorem 0.5

If function $f\geq 0$ over $[a,b]$, then

$$\int_a^b f(x)\,dx\geq 0$$

and the equality holds if and only if $f(x)\equiv 0$.

Proof: Since we know that

$$\int_a^b f(x)\,dx\geq 0=\lim_{||T||\to 0}\underline{S}(T)=\lim_{||T||\to 0}\sum_{i=1} m_i\Delta x_i$$

where we know that both m_i and Δx_i are greater or equal to 0, thus the whole thing ≥ 0. □

Corolary 0.1

As a consequence, suppose f and g are continous, we have

$$\left(\int_a^b fg\,dx\right)^2\leq\left(\int_a^b f^2\,dx\right)\left(\int_a^b g^2\,dx\right)$$

Proof: We know that

$$\int_a^b (f+tg)^2\,dx\geq 0$$

Hence we define

$$G(t) \ := \ t^2\int_a^b g^2 \ dx + 2t\int_a^b fg \ dx + \int_a^b f^2 \ dx \geq 0$$

which is essentially a parabola, so we consider the minimum point of the porabola:

$$G'(t) = 0 \quad \Rightarrow \quad t_0 = -\frac{\int_a^b fg \ dx}{\int_a^b g^2 \ dx}$$

Therefore, pluging t_0 into $G(t)$ we can obtain that $G(t_0) \geq 0$, which gives us the result. □

Theorem 0.6: Fundamental T of Calculus

We have

$$\frac{d}{dx}\int_a^x f(t) \ dt = f(x)$$

Proof: Notice that

$$\begin{aligned} LHS &= \lim_{h\to 0}\frac{\int_a^{x+h} f(t) \ dt - \int_a^x f(t) \ dt}{h} \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t) \ dt}{h} \end{aligned}$$

Therefore we can obtain that

$$LHS \ - \ RHS = \lim_{h\to 0}\frac{\int_x^{x+h} f(t) - f(x) \ dt}{h}$$

Since f is continous at x, then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|t-x| < \delta$, we have $|f(t)-f(x) < \epsilon|$, this yields us that

$$\begin{aligned} \lim_{h\to 0}\left|\frac{\int_x^{x+h} f(t) - f(x) \ dt}{h}\right| &\leq \lim_{h\to 0}\frac{\int_x^{x+h} |f(t) - f(x)| \ dt}{|h|} \\ &< \frac{\int_x^{x+h} \epsilon \ dt}{|h|} = \epsilon \end{aligned}$$

as desired. □

Result 0.4

Why do we need such an abstract consequence?

Example 0.18

Suppose we want to find

$$F(x) = \int_0^x e^{-t^2} \ dt$$

We now know that

$$\begin{cases} \frac{dF}{dx} = e^{-x^2} \\ F(0) = 0 \end{cases}$$

The question we solved before (HW1 & HW2) all have explicit solutions, but for this one, we cannot find a direct solution. The above result suggests that there exists a theoreticla solution.

Exercise:

$$\begin{aligned} \frac{dF}{dx} &= e^{-x^2} = \frac{n}{0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \\ \Rightarrow \quad F &= \frac{n}{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \end{aligned}$$

Example 0.19

Evaluate

$$\int_0^{\infty} e^{-x^2}\ dx$$

Proof:

$$I(t) = \int_0^{\infty} \frac{e^{-x^2}}{1+\left(\frac{x}{t}\right)^2}\ dx$$

Hence our goal is $I(\infty)$

$$\begin{aligned} I(t) &= t\int_0^{\infty} \frac{e^{\left(\frac{x}{t}\right)^2 t^2}}{1+\left(\frac{x}{t}\right)^2}\ d\left(\frac{x}{t}\right) \\ &= t\int_0^{\infty} \frac{e^{-t^2x^2}}{1+x^2}\ dx \end{aligned}$$

Notice that we have $I(0) = 0$, and $\lim\limits_{t\to 0} \frac{I(t)}{t} = \frac{\pi}{2}$, thus

$$\begin{aligned} t^{-1}I(t)e^{-t^2} &= \int_0^{\infty} \frac{e^{-t^2(x^2+1)}}{1+x^2}\ dx \\ \Rightarrow \quad \frac{d}{dx}\left(t^{-1}I(t)e^{-t^2}\right) &= -2t\int_0^{\infty} e^{-t^2(x^1+1)}\ dx \\ &= -2e^{-t^2}I(\infty) \\ \Rightarrow \quad \int_0^{\infty} \frac{d}{dx}\left(t^{-1}I(t)e^{-t^2}\right) &= -2\int_0^{\infty} e^{-t^2(x^2+1)}\ dtI(\infty) \\ \Rightarrow \quad -\lim_{t\to 0} \frac{e^{-t^2}I(t)}{t} &= -2I(\infty)^2 \\ \Rightarrow \quad -\frac{\pi}{2} &= -2I(\infty)^2 \end{aligned}$$

which yields us that $I(\infty) = \frac{\sqrt{\pi}}{2}$, thus implying that

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

as desired. $\square$

Lec 10 - Mon - Jan 29 - 2024

Goal 0.3

There is another approach of computing $\int_0^\infty e^{-x^2}\, dx$ by considering $\int_{-\infty}^\infty e^{-x^2}\, dx \cdot \int_{-\infty}^\infty e^{-t^2}\, dt$. The intuition is thinking about the diagram in three dimensional space, and we are calculating the volumn of the shape. We will learn how to deal with this in the future.

Substitution

Theorem 0.7: Substitution

Remark: Recall FTC2 a bit from Shum's section.

We have

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f \circ \varphi(t) \cdot \varphi'(t)\, dt$$

where $\varphi([\alpha, \beta]) = [a, b]$ and φ' exists (idealy continuous).

Proof: Notice that $LHS = \int_a^{\varphi(\beta)} f(x)\, dx$, thus

$$\frac{d}{d\beta} LHS = f(\varphi(\beta)) \cdot \varphi'(\beta)$$
$$\frac{d}{d\beta} RHS = f \circ \varphi(\beta) \cdot \varphi'(\beta)$$

since $\varphi([\alpha, \beta]) = [a, b]$. Hence $LHS = RHS$. $\square$

Integration by Parts

Theorem 0.8: Integration by Parts

We have

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

Proof: Notice that

$$\begin{aligned}\frac{d}{db}\ LHS &= u(b)v'(b)\\ \frac{d}{db}\ RHS &= u'(b)v(b) + u(b)v'(b) + v(b)u'(b)\\ &= u(b)v'(b)\end{aligned}$$

Hence $LHS = RHS$. $\square$

Example 0.20

Find $\displaystyle\int_0^1 \frac{\arctan x}{1+x}\ dx$

Proof: We can find that

$$\int_0^1 \frac{\arctan x}{1+x}\ dx = \int_0^1 \frac{\arctan x}{1+x}\ d\ln(1+x) = \arctan x \ln|1+x|\Big|_0^1 - \int_0^1 \frac{\ln(1+x)}{1+x^2}\ dx$$

Thus the point of the question is we want to evaluate the latter part, to evaluate, we have the following two methods:

1. Method 1

   Let $x = \tan\theta$, thus we have

$$\begin{aligned}\int_0^1 \frac{\ln(1+x)}{1+x^2}\ dx &= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\ d\tan\theta\\ &= \int_0^{\pi/4} \ln(1+\tan\theta)\ d\theta\\ &= \int_0^{\pi/4} \ln(\sin\theta+\cos\theta)\ d\theta - \int_0^{\pi/4} \ln\cos\theta\ d\theta\\ &= \int_0^{\pi/4} \ln\left(\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\right)\ d\theta - \int_0^{\pi/4} \ln\cos\theta\ d\theta\\ &= \int_0^{\pi/4} \ln\sqrt{2}\ d\theta + \underbrace{\int_0^{\pi/4} \ln\left(\sin\left(\theta+\frac{\pi}{4}\right)\right)\ d\theta - \int_0^{\pi/4} \ln\cos\theta\ d\theta}_{0}\end{aligned}$$

2. Method 2

   Let $\displaystyle\int_0^1 \frac{\ln(1+tx)}{1+x^2}\ dx$, notice that $I(0) = 0$ and out goal is $I(1)$.

$$\begin{aligned}I'(t) &= \int_0^1 \frac{x}{(1+x^2)(1+tx)}\ dx\\ &= \frac{1}{1+t^2}\int_0^1 \frac{t}{1+x^2} + \frac{x}{1+x^2} - \frac{t}{1+tx}\ dx\\ &= \frac{\pi}{4}\frac{1}{1+t^2} + \frac{\ln 2}{2}\frac{1}{1+t^2} - \frac{\ln(1+t)}{1+t^2}\end{aligned}$$

as desired. $\square$

Lec 10.1 (Shum section) - Mon - Jan 29 - 2024

Substitution

Theorem 0.9: Substitution rule for definite integrals

Let $u : [a, b] \to I$ be differentiable with continuous derivative on $[a, b]$ and let f be continuous on I, then

$$\int_a^b f(u(x))\frac{du}{dx}\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Proof:

$$\frac{d}{dx}F(u(x)) = \frac{d}{du}F(u)\Big|_{u=u(x)} \cdot \frac{d}{dx}u\Big|_x = f(u(x)) \cdot \frac{du}{dx}(x)$$

By FTC1, we have

$$\int_{u(a)}^{u(b)} \underbrace{\frac{d}{du}F(u)}_{f(u)}\,du = F(u(b)) - F(u(a))$$

Also, by FTC1,

$$\int_a^b f(u(x))\,\frac{du}{dx}\,dx = G(b) - G(a)$$

where G is the antiderivative of $f(u(x))\dfrac{du}{dx}$. We can take $G = F \circ u$, then

$$\int_a^b f(u(x))\frac{du}{dx}\,dx = F(u(b)) - F(u(a))$$

as desired. $\square$

Example 0.21

Find $\displaystyle\int_0^2 \frac{1}{2x+1}\,dx$.

Proof: Let $u = 2x + 1$, then $\dfrac{du}{dx} = 2$, thus

$$\begin{aligned}\int_0^2 \frac{1}{2x+1}\,dx &= \int_0^2 \frac{1}{u(x)} \cdot \frac{1}{2}\,\frac{du}{dx}\,dx \\ &= \frac{1}{2}\int_1^5 \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u|\Big|_1^5 = \frac{1}{2}\ln 5\end{aligned}$$

as desired. □

Proposition 0.3

Let f be continuous on $[-a, a]$, for some $a > 0$

1. If f is even, then
$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

2. If f is odd, then
$$\int_{-a}^{a} f(x)\, dx = 0$$

Proof: for part 1 By addition property, we have

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

By even property,

$$\int_{-a}^{0} f(x)\, dx = \int_{-a}^{0} f(-x)$$

Use substitution with $u = -x$, then we would obtain that result. □

Inverse Trigonometric Substitution

Example 0.22

Find $\displaystyle\int x\sqrt{1-x^2}\, dx$ and $\displaystyle\int \sqrt{1-x^2}\, dx$
Example: This is ann example which is impossible to solve if use the conventional substitution method.

Example 0.23

Find $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\, dx$

Proof: let $x = \sin u$, thus we have

$$\begin{aligned}\int_{-1}^{1} \sqrt{1-x^2}\, dx &= \int_{-\pi/2}^{\pi/2} \cos u \cos u\, du \\ &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(1+\cos 2u\right)\, du \\ &= \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right)\Big|_{-\pi/2}^{\pi/2}\end{aligned}$$

Thus the final answer is $\frac{\pi}{2}$. $\square$

Tutorial (Shum section) - Mon - Jan 29 - 2024

Example 0.24

Find $\int_0^1 \frac{x-1}{\ln x}\, dx$

Proof: Feynman's Trick (Leibniz integral rule):

$$\frac{d}{dt}\int_a^b f(x,t)\, dx = \int_a^b \frac{\partial}{\partial t} f(x,t)\, dx$$

We let

$$I(t) = \int_0^1 \frac{x^t - 1}{\ln x}\, dx$$

Thus our goal is to find $I = I(1) - I(0)$, now we have

$$\begin{aligned}\frac{\partial}{\partial t} I(t) &= \frac{\partial}{\partial t}\int_0^1 \frac{x^t - 1}{\ln x}\, dx \\ &= \int_0^1 x^t\, dx = \frac{x^{t+1}}{t+1}\Big|_0^1 = \frac{1}{t+1}\end{aligned}$$

Therefore, recall the FTC and what we have written down, we have

$$\begin{aligned}I = I(1) - I(0) &= \int_0^1 I'(t)\, dt \\ &= \int_0^1 \frac{1}{t+1}\, dt\end{aligned}$$

which is simple to evaluate. $\square$

Tutorial (Chen section) - Mon - Jan 29 - 2024

Problem set

1. Let $f(x) = x^2$ on $[0,1]$, let $T_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$

   (a) Compute $\overline{S}(T_n)$

   Proof: We can find that

$$\begin{aligned}\overline{S}(T_n) &= \sum_{i=1}^{n} \frac{1}{n} \cdot \left(\frac{i}{n}\right)^2 \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+1)(2n+1)}{6n^2}\end{aligned}$$

as desired. $\square$

(b) Compute $\underline{S}(T_n)$

Proof: Similarly, we can find that

$$\underline{S}(T_n) = \overline{S}(T_n) - \frac{1}{n}$$

I think this is right? $\square$

(c) Use this to show that $f(x)$ is Riemann integrable and computer $\displaystyle\int_0^1 x^2\, dx$

Proof:

$\square$

HINT: You may want to use $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

2. Let

$$f(x) = \begin{cases} \frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

Is $f(x)$ Riemann integrable on $[a,b]$ or not? Why or why not?

3. Show that

$$\int_0^a e^{-x^2}\, dx \leq 1 + e^{-1} - e^{-a}$$

CHALLENGE: Can you find a better bound?

4. We know that continuous functions are Riemann integrable, but the inverse is not true.

(a) Find a discontinuous function which is Riemann integrable, prove by showing lowver sum is equal to the upper sum.

(b) TRICKY! Find a Riemann-integrable function with infinitely many discontinuities.

Proof: Consider

$$f(x) = \begin{cases} 1 & x = \frac{1}{n} \text{ for } n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

as desired. $\square$

(c) Challenge: Find a function with uncountably many discontinuities.

Recall from last lecture, we wanted to evaluate $\int_0^1 \frac{\ln(1+x)}{a+x^2}\,dx$, and we can parameterize it by defining

Example 0.25

parameterize

$$I(t) = \int_0^1 \frac{\ln(1+tx)}{a+x^2}\,dx$$

Notice that $I(0) = 0$ and our goal is $I(1)$.

How accurate is Taylor Expansion?

Suppose we have functino f and first order Taylor polynomial, then

$$|f - f_1| = \mathcal{O}\left((x-x_0)^2\right)$$

We will show that the difference is bounded by $M(x-x_0)^2$ if $|f''(x)| \leq M$.
Remark: This is an application of the Fundamental Theorem of Calculus.

Suppose function $f(x)$, then the first order taylor polynomial of f at point a is

$$f_1(x) = f(a) + f'(a)(x-a)$$

and we define the error by

$$R = f(x) - f(a) + f'(a)(x-a)$$

Goal 0.4

The key idea for this is to view R as a function of a while x is fixed instead.

Therefore, we have

$$R(a) = f(x) - f(a) + f'(a)(x-a)$$

Hence we have

$$\begin{cases} R(x) = 0 \\ \frac{dR}{da} = -f'(a) + f''(a)(x-a) + f'(a) = f''(a)(x-a) \end{cases}$$

Therefore we have

$$\begin{aligned} R(a) &= \int_x^a \frac{dR}{d\tilde{a}}\,d\tilde{a} \\ &= \int_x^a f''(\tilde{a})(x-\tilde{a})\,d\tilde{a} \end{aligned}$$

Goal 0.5

Our goal (claim) is

$$R(a) = -f''(\xi) \int_x^a (x - \tilde{a})\, d\tilde{a}$$

for some ξ between x and a.

Proof: As a consequence of the IVT, for some ξ between x and a:

$$\frac{\int_x^a -f''(\tilde{a}) \cdot 1\, d\tilde{a}}{\int_x^a 1\, d\tilde{a}} = -f''(\xi)$$

because we know that

$$\frac{\int_x^a -f''(\tilde{a}) \cdot 1\, d\tilde{a}}{\int_x^a 1\, d\tilde{a}} \quad \in \quad \left[\inf_{\alpha \in [x,a]} -f''(\tilde{a}), \sup_{\alpha \in [x,a]} -f''(\tilde{a})\right]$$

Reasonable eh? □

Result 0.5

Therefore we have

$$R(a) = \frac{f''(\xi)(a - x)^2}{2}$$

Therefore,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(\xi)(a - x)^2}{2}(x - a)^2$$

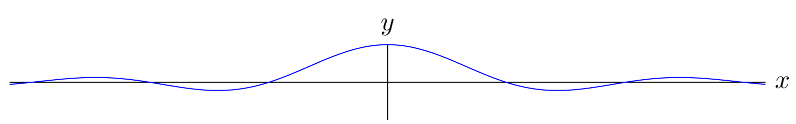
for $\xi \in [x, a]$.

Exercise: Do the same process for the 2^{nd} order Taylor polynomial.

Example 0.26

What is the meaning for

$$\int_0^\infty \frac{\sin x}{x}\, dx$$



Example: Suppose instead we have

$$\lim_{t \to \infty} \int_0^t \frac{\sin x}{x}\, dx$$

if t is finite, then it is well-defined, the key reason for this is because this oscillates.

Example 0.27

We first consider a special case:

$$\lim_{n\to\infty}\int_0^{2n\pi}\frac{\sin x}{x}\,dx$$

Proof: Notice that the sum is bounded by 0 and a_1, and the area keeps increasing as $n\to\infty$. $\square$

Remark: Oopsies, time's up for Chen.

Lec 12 - Fri - Feb 2 - 2024

Goal 0.6

We want to find

$$\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}$$

and make it rigorous.

What is the meaning?

$$\int_0^{\infty}\frac{\sin x}{x}\,dx=\lim_{t\to\infty}\int_0^{t}\frac{\sin x}{x}\,dx$$

the reason why we introduce the limit is because infinite partition does not make sense (not defined).

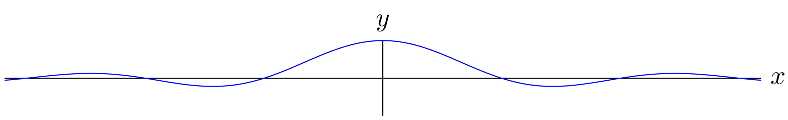
Why does the limit exist?

Example 0.28: Counter Example

$$\int_1^{\infty}\frac{1}{x}\,dx=\lim_{t\to\infty}\int_1^{t}\frac{1}{x}\,dx=\lim_{t\to\infty}\ln t=\infty$$

As an counterexample, the above does not exist as $\mathbb{R}$. However, in our case, we can define

$$S_{2n}=\int_0^{2n\pi}\frac{\sin x}{x}\,dx$$



Refering to the diagramm above, we denote each area of the region as $\{a_i\}_{i\in\mathbb{N}}$, thus we have

$$S_{2n}=a_1+(-a_2+a_3)+(-a_4+a_5)+\cdots<a_1$$
$$S_{2n+2}=S_{2n}+(-a_{2n+1}+a_{2n+2})>S_{2n}$$

(note that we can group them because we have finite number of terms). As a result, we have

$$\lim_{n\to\infty} S_{2n} \quad \text{exists}$$

General Case

Goal 0.7

Claim:

$$\lim_{t\to\infty}\int_0^t \frac{\sin x}{x}\,dx$$

exists.

Proof: We have

$$\int_0^t \frac{\sin x}{x}\,dx = S_{2(n-1)\pi} + \int_{2(n-1)\pi}^t \frac{\sin x}{x}\,dx$$

Suppose $t \in (2(n-1)\pi, 2n\pi)$, notice that

$$\begin{aligned}\int_{2(n-1)\pi}^t \frac{\sin x}{x}\,dx &\le \frac{1}{2(n-1)\pi}\int_{2(n-1)\pi}^t \sin x\,dx \\ &\le \frac{1}{2(n-1)\pi}\underbrace{\int_{2(n-1)\pi}^{2n\pi} \sin x\,dx}_{\to 0}\end{aligned}$$

which implies that the integral exists. □

Remark: On the other hand, we here provide another alternative proof.

Proof: We have

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \lim_{t\to\infty}\int_1^t \frac{\sin x}{x}\,dx$$

Note that

$$\begin{aligned}\int_1^t \frac{\sin x}{x}\,dx &= -\int_1^t \frac{1}{x}\,d\cos x \\ &= -\frac{\cos x}{x}\Big|_1^t - \int_1^t \frac{\cos x}{x^2}\,dx\end{aligned}$$

Now the question becomes:

Whether $\lim_{t\to 0}\int_0^t \frac{\cos x}{x^2}\,dx$ exists or not?

Proof: SFAC it does not exist, then $\exists\ \epsilon_0 > 0$ such that $\forall\ i$, $\exists\ s_i \ge t_i \ge i$ such that

$$\left|\int_i^{s_i} \frac{\sin x}{x^2}\,dx - \int_1^{t_i} \frac{\sin x}{x^2}\,dx\right| > \epsilon_0$$

We can easily find out that

$$
\begin{aligned}
LHS &= \left| \int_{t_i}^{s_i} \frac{\cos x}{x^2}\, dx \right| \\
&\leq \int_{t_i}^{s_i} \frac{1}{x^2}\, dx \\
&= -\frac{1}{x}\Big|_{t_1}^{s_i} \\
&= -\frac{1}{s_i} + \frac{1}{t_i} \\
&< \frac{1}{t_i}
\end{aligned}
$$

which contradicting since we have $0 \geq \epsilon_0 > 0$. □

as desired. □

Result 0.6

Here we have two results:

(a) Suppose $a_i \to 0$, then

$$\lim_{n\to\infty} \sum_{i=0}^{n} (-1)^i a_i$$

exists.

(b) Suppose $|g| \leq |f|$ (continuous), if

$$\int_0^\infty |f|\, dx$$

exists, then

$$\int_0^\infty |g|\, dx$$

exists.

Recall Feynmann's Trick, we had

$$I(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx}\, dx \qquad t \geq 0$$

Our goal is to find $I(0)$, we have

$$\begin{aligned}
I'(t) &= \int_0^\infty -\sin x e^{-tx}\ dx = -\frac{1}{1+t^2} \qquad t>0 \\
\Rightarrow \quad I(0) &= \int_\infty^0 I'(t)\ dt + I(\infty) \\
&= \int_\infty^0 -\frac{1}{1+t^2}\ dx + I(\infty) \\
&= \frac{\pi}{2}
\end{aligned}$$

In this process, we assumed that

$$\lim_{t\to 0^+} I(t) = I(0)$$

but we wonder why is this true?

Example 0.29

$$\lim_{t\to 0^+}\int_0^\infty \frac{\sin x}{x} e^{-tx}\ dx = \int_0^\infty \frac{\sin x}{x} \lim_{t\to 0^+} e^{-tx}\ dx$$

Goal 0.8

We want to show

$$\lim_{t\to 0^+}\int_0^\infty \frac{\sin x}{x}\left(e^{-tx}-1\right)\ dx$$

is small.

Proof: We have

$$\lim_{t\to 0^+}\int_0^\infty \frac{\sin x}{x}\left(e^{-tx}-1\right)\ dx = \underbrace{\int_0^N \frac{\sin x}{x}\left(e^{-tx}-1\right)\ dx}_{A} + \underbrace{\int_N^\infty \frac{\sin x}{x}\left(e^{-tx}-1\right)\ dx}_{B}$$

Hence we choose large enough N, we have $|B| < \frac{\epsilon}{2}$, and we can choose small enough t such that $|A| < \frac{\epsilon}{2}$. □

Here we provide an alternative proof:

Proof: We have

$$\begin{aligned}
\left|\int_N^\infty \frac{\sin x}{x}\left(e^{-tx}-1\right)\right| &= \left| -\int_N^\infty \frac{e^{-tx}-1}{x}\ d\cos x\right| \\
&= \left| \underbrace{-\frac{e^{-tx}-1}{x}\cos x\Big|_N^\infty}_{\text{as N}\to\infty\text{, =0}} + \underbrace{\int_N^\infty \cos\ d\left(\frac{e^{-tx}-1}{x}\right)}_{\text{as N}\to\infty\text{, =0}} \right|
\end{aligned}$$

as desired. □

Goal 0.9

Recall from the last example, we want to know that

(a) $I(0) = I(0^+) = \lim_{t\to 0^+} I(t)$

(b)
$$\frac{d}{dt}\int_0^\infty \frac{\sin x}{x} e^{-tx}\,dx = \int_0^\infty \frac{d}{dt}\frac{\sin x}{x} e^{-tx}\,dx$$

Proof: of (a)
We want to know that
$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{t\to 0^+}\int_0^\infty \frac{\sin x}{x} e^{-tx}\,dx$$
Hence it suffices to prove that
$$\int_0^\infty \frac{\sin x}{x} e^{-tx}\,dx = \underbrace{\int_0^N \frac{\sin x}{x}(e^{-tx}-1)\,dx}_{A} + \underbrace{\int_N^\infty \frac{\sin x}{x}(e^{-tx}-1)\,dx}_{B} = 0$$

(i) Step 1: Choose N large, so that $|B| < \frac{\varepsilon}{2}$

$$\begin{aligned} |B| &= -\int_0^\infty \frac{e^{-tx}-1}{x}\,d\cos x \\ &= \frac{e^{-tN}-1}{N}\cos(N) + \int_N^\infty \cos x \frac{d}{dx}\left(\frac{e^{-tx}-1}{x}\right)\,dx \end{aligned}$$

where we can make both expression smaller than $\frac{\varepsilon}{4}$, which is small enough for us to prove our argument.

(ii) Step 2: Fix N, choose δ so that if $0 < t < \delta$, then $|A| < \frac{\varepsilon}{2}$

CLAIM: $\exists\,\delta$ such that if $0 < t < \delta$, then $\left|e^{tx}-1\right| < \frac{\varepsilon}{2M_0}$ for all $x \in [0, N]$, where $M_0 = N \cdot \sup_{[0,N]}\left|\frac{\sin x}{x}\right|$.

As a result of the claim, we have

$$\begin{aligned} \left|\int_0^N \frac{\sin x}{x}\left(e^{-tx}-1\right)\,dx\right| &\le \int_0^N \left|\frac{\sin x}{x}\right| \frac{\varepsilon}{2M_0}\,dx \\ &\le N\sum_{[0,N]}\left|\frac{\sin x}{x}\right|\frac{\varepsilon}{2M_0} \\ &= \frac{\varepsilon}{2} \end{aligned}$$

Thus it suffices to prove the claim:

SFAC that for all $\frac{1}{i}$, $\exists\, 0 < t_i < \delta_i$, but

$$\left|e^{t_i x_i} - 1\right| \geq \frac{\varepsilon}{2M_0}$$

for some $x_i \in [0, N]$. However, since we know that x_i exists in a closed interval and therefore as $t \to 0$, we have $t_i x_i \to 0$, and this implies that

$$0 = \left|e^{-t_i x_i} - 1\right| \geq \frac{\varepsilon}{2M_0} > 0$$

which is a contradiction.

thus we complete the prove. $\square$

Proof: of (b)
we know

$$\frac{d}{dx}\int_0^\infty \frac{\sin x}{x} e^{-tx}\, dx = \lim_{h\to 0}\int_0^\infty \frac{\sin x}{x}\frac{e^{-x(t+h)} - e^{-tx}}{h}\, dx$$

and we have already shown that

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{d}{dx}(e^{-tx})\, dx$$

exists, so it suffices to prove that

$$\lim_{h\to 0}\int_0^\infty \frac{\sin x}{x}\left(\frac{e^{-x(t+h)} - e^{-tx}}{h} - (-xe^{-tx})\right)\, dx = 0$$

Remark: Repeat the prove in part (a)

GG $\square$

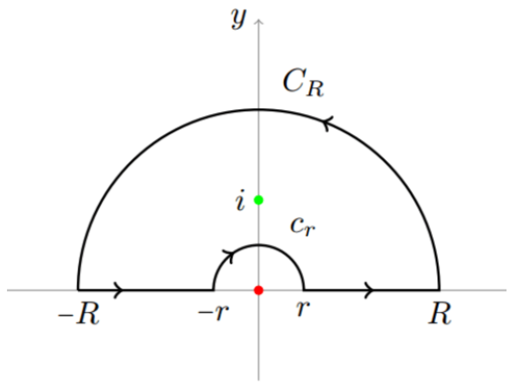
Lec 14 - Wed - Feb 7 - 2024

A Computation (wont be tested)

Notice that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \frac{e^{ix}}{x}\, dx$$

We can view $\frac{e^{ix}}{x}$ as a function over $\mathbb{C}$.



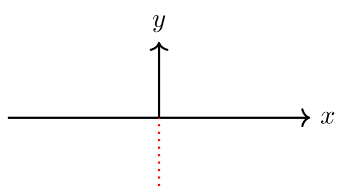
Result 0.7

We have

$$\int_{C_{r,R}} \frac{e^{iz}}{z}\, dz \;\equiv\; 0$$

(the intuition here is the displacement of you walking from point R to point R, which is essentially 0).

Proof: There exists F over



such that

$$\begin{aligned} \frac{dF}{dz} &= \frac{e^{iz}}{z} \\ \Rightarrow \quad \int_{C_{r,R}} \frac{e^{iz}}{z}\, dz &= \int_{C_{r,R}} F'(z)\, \frac{dz}{dy}\, dt \\ &= \int_{C_{r,R}} \frac{d\, F(z)}{dt}\, dt \\ &= F(R) - F(R) = 0 \end{aligned}$$

Therefore, try to evaluate the integral from 0 to ∞, we would like to have $r \to 0^+$ and $R \to \infty$. Thus

$$\int_{C_\infty} + \int_{C_{0+}} + \int_{\infty}^{0^-} + \int_0^{\infty} = 0$$

Therefore we have

$$\begin{aligned}\int_{C_R} \frac{e^{iz}}{z}\,dz &= \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\,dRe^{i\theta}\\ &= \int_0^\pi ie^{i(R(\cos\theta + i\sin\theta))}\,d\theta\\ &= i\int_0^\pi e^{-R\sin\theta}e^{iR\cos\theta}\,d\theta\end{aligned}$$

Therefore we have that

$$\left|\int_{C_{r,R}} \frac{e^{iz}}{z}\,dz\right| \leq \int_0^\pi e^{-R\sin\theta}\,d\theta$$

and we know that the RHS approaches 0 as R approaches 0, thus completing the proof. □

Lec 15 - Fri - Feb 8 - 2024

Goal 0.10

We want to explain why

$$\sin x \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Proof: By the taylor remainder theorem, we know that

$$\sin x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \frac{\sin^{(2n+3)}(\xi)}{(2n+3)!}(x-a)^{2n+3}$$

and easy to find that the RHS is approaching 0 as $n \to \infty$. □

Goal 0.11

What if we dont know that the infinite sum is equal to $\sin x$?

Note that for x fixed, we have that

$$\frac{(-1)^n x^{2n+1}}{(2n+1)!} < \frac{1}{2^n}$$

for large n.

Result 0.8

Our claim is

$$S_n = \sum_{n=0}^{N} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

converges as $N \to \infty$.

Proof: SFAC that this is not true, thus we would have that $\exists\, \varepsilon_0 > 0$ such that $\forall\, N$, $\exists\, A_N \leq B_N \leq N$ such that

$$\Big|S_{A_N} - S_{B_N}\Big| \geq \varepsilon_0 > 0$$

Notice that for the LHS, we have

$$\begin{aligned}\Big|S_{A_N} - S_{B_N}\Big| &= \sum_{n=B_N}^{A_N} \Big|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\Big| \\ &\leq \sum_{n=B_N}^{A_N} \frac{1}{2^n}\frac{1}{2^n} \\ &< \frac{1}{2^{B_N - 1}}\end{aligned}$$

Similarly, we know that

$$\sum_{n=0}^{\infty} \frac{d}{dx}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

also converges.

Therefore, we want to show that

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{d}{dx}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Recall the definition of derivative, STP that

$$\lim_{h\to 0}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{(x+h)^{2n+1} - x^{2n+1}}{h} - (2n+1)x^{2n}\right) = 0$$

We can split the summation into two parts as shown following:

$$\sum_{0}^{\infty} = \sum_{0}^{N} + \sum_{N+1}^{\infty}$$

Therefore, we can choose large enough N so that $\displaystyle\sum_{N+1}^{\infty} < \frac{\varepsilon}{2}$, and for such fixed N, we can find small enough h such that $\displaystyle\sum_{0}^{N}$ is also smaller than $\dfrac{\varepsilon}{2}$, thus we have completed the proof. □

Lec 16 - Mon - Feb 12 - 2024

Recall from last lecture, we have that

$$\frac{1}{x-1} = \sum_{n=0}^{\infty} x^n \qquad \text{for} \quad |\text{x}| < 1$$

Goal 0.12

Our goal is to show that

$$\frac{d}{dx}\,\frac{1}{x-1} = \sum_{n=0}^{\infty} \frac{d}{dx}\, x^n$$

To do so, we need to show

1. The RHS is well-defined
2. Commutativity of the derivative

We first prove the first goal, that is, proving that the RHS is well-defined.

Theorem 0.10

Here comes the trick:

$$a_n := \left|nx^{n-1}\right|$$

Therefore we can have

$$\begin{aligned}
\lim_{n\to\infty} a_n{}^{\frac{1}{n}} &= \lim_{n\to\infty} n^{\frac{1}{n}}|x|^{1-\frac{1}{n}} \\
&= \lim_{n\to\infty} e^{\frac{1}{n}\ln n}|x|^{1-\frac{1}{n}} \\
&= |x| \\
\Rightarrow \quad a_n &\leq \left(\frac{|x|+1}{2}\right)^n \qquad \text{for } n \text{ large}
\end{aligned}$$

Proof: We can find that

$$\lim_{n\to\infty} \frac{a_n}{\left(\frac{|x|+1}{2}\right)^n} = \lim_{n\to\infty} \left(\frac{a_n{}^{\frac{1}{n}}}{|x|} \cdot \frac{|x|}{\frac{|x|+1}{2}}\right)^n$$

Notice that for large n we would have $\dfrac{a_n{}^{\frac{1}{n}}}{|x|} = 1$ and $\dfrac{|x|}{\frac{|x|+1}{2}} < 1$, which yields us that

$$\lim_{n\to\infty} \frac{a_n}{\left(\frac{|x|+1}{2}\right)^n} = 0$$

thus proven convergence. □

Goal 0.13

Now STP commutativity.

Proof: We have

$$\frac{d}{dx}\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\frac{d}{dx}x^n=\lim_{h\to 0}\sum_{n=0}^{\infty}\left(\frac{(x+h)^n-x^n}{h}-nx^{n-1}\right)$$
$$=\underbrace{\lim_{h\to 0}\sum_{n=0}^{N-1}}_{A}+\underbrace{\lim_{h\to 0}\sum_{n=N}^{\infty}}_{B}$$

Hence we can choose N large so that $|B|<\frac{\epsilon}{2}$, and then we can choose δ' so that for all $|h|<\delta'$, we have $|A|<\frac{\epsilon}{2}$. □

Remark: We also need a range for h in part B to make smaller than $\frac{\epsilon}{2}$, thus, we need to choose $\delta=\min\{\delta_A,\delta_B\}$.

Proof: Notice that by mean value theorem, we have

$$\sum_{n=N}^{\infty}\left(\frac{(x+h)^n-x^n}{h}-nx^{n-1}\right)$$
$$=\sum_{n=N}^{\infty}\left(n(x+\xi_n h)^{n-1}-nx^{n-1}\right)\qquad\text{for}\quad \xi_{\mathrm{n}}\in[0,1]$$

gg. □

Result 0.9: Summary of Tricks

To wrap up, we have

1. For $\sum_{n=0}^{\infty}a_n$, if $|a_n|^{\frac{1}{n}}\leq r\leq 1$ for large n, then $\sum_{n=0}^{\infty}a_n$ converges.
2. For $\sum_{n=0}^{\infty}a_n$, if $\left|\frac{a_{n+1}}{a_n}\right|\leq r\leq 1$ for large n, then $\sum_{n=0}^{\infty}a_n$ converges.

Definition 0.10: Dirichlet's Test

We want to analyze $\sum_{n=1}^{N} a_n b_n$. Suppose $S_n := \sum_{n=1}^{N} b_n$, thus we have

$$\begin{aligned}\sum_{n=1}^{N} a_n b_n &= \sum_{n=1}^{N} a_n \left(S_n - S_{n-1}\right)\\ &= \sum_{n=1}^{N} a_n S_n - \sum_{n=1}^{N} a_n S_{n-1}\\ &= \sum_{n=1}^{N} a_n S_n - \sum_{n=1}^{N-1} a_{n+1} S_n\\ &= \sum_{n=1}^{N} (a_n - a_{n+1}) S_n + a_N S_N\end{aligned}$$

if $|S_n| \leq M < \infty$ and $a_n \to 0$, then we have

$$\sum_{n=1}^{N} a_n b_n \qquad \text{converges}$$

Definition 0.11: Abel

Suppose $b_n \to b$ and $\sum_{n=1}^{\infty} a_n$ is convergent, then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n (b_n - b) + \sum_{n=1}^{\infty} b a_n$$

is also convergent.

Lec 17 - Wed - Feb 14 - 2024

Goal 0.14

If $f_n(x) \rightrightarrows f(x)$ and $f_n(x)$ is continuous, is $f(x)$ continuous.

Example 0.30: Counter Example

One counter example would be $f_n(x) = x^n$, then $\lim_{n\to\infty} x^n$ is not continuous since over $[0,1]$,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & x \neq 1 \end{cases}$$

Consider the following reasoning:

Proof: We want to have $f(x)$ to be continuous at x_0: $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ |x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.

Since we know that $f_n(x)$ is continuous, so

$$|f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

and

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Step 1: for x fixed, we choose n so that $|f_n(x) - f(x)| < \frac{\epsilon}{3}$.

Step 2: for n fixed, we choose δ so that if $|x - x_0| < \delta$, we have $|f_n(x) - f(x_0)| < \frac{\epsilon}{3}$.

However, this process is wrong since we choose x to find δ, so x is dependent on δ now. $\square$

Definition 0.12: Uniformly Convergent

f_n is uniformly convergent to f over $[a, b]$ if for any $\epsilon > 0$, there exists N so that if $n > N$,

$$|f_n(x) - f(x)| < \epsilon$$

for any $x \in [a, b]$. We $f_n \rightrightarrows f$.

Result 0.10

If f_n is uniformly convergent to f, then f_n continuous implies that f is also continuous.

Example 0.31

$\sin\left(\frac{x}{n}\right) \ \rightrightarrows\ 0$ over $[-1, 1]$.

Example 0.32

We have $x^n \not\rightrightarrows 0$ over $[0, 1)$, since we know

$$x^n \to 1 \qquad \text{as} \qquad x \to 1^-$$

Example 0.33

We have $nx \sin\left(\frac{1}{nx}\right) \not\rightrightarrows 1$ over $[-1, 1]$, since we have

$$n \cdot \frac{2}{\pi n} \sin\left(\frac{1}{n \cdot \frac{2}{\pi n}}\right) = \frac{2}{\pi}$$

Theorem 0.11

If $f_n \to f$ over $[a,b]$ and f continuous $f_n(x_n) = f(x)$ for any $x_n \to x \iff f_n \rightrightarrows f$ over $[a,b]$ and f is continuous.

Proof:

1. $(\Longleftarrow)$
   We have
   $$f_n(x) - f(x) = \underbrace{f_n(x_n) - f_n(x)}_{=f_n'(\xi_n)(x_n - x)} + \underbrace{f_n(x) - f(x)}_{\to 0 \text{ by assumption}}$$
2. $(\Longrightarrow)$
   Follow the above examples and steps.

thus completing the proof. □

Theorem 0.12: Arzela-Ascolli

If $|f_n| \leq C \ll \infty$, then $f_n \to f$ over $[a,b]$ implies $f_n \rightrightarrows f$.

Lec 18 - Fri - Feb 16 - 2024

Recall from last lecture, for $f_n(x)$ and $f(x)$ continuous over $[a,b]$. If $f_n(x) \not\rightrightarrows f(x)$, then there exists $x_n \to x \in [a,b]$ such that $f_n(x) \not\to f(x)$.

Goal 0.15

Suppose $f_n \to f$, is it true that
$$\int_a^b f_n \, dx \to \int_a^b f \, dx$$

1. Consider
   $$nx \sin\left(\frac{1}{nx}\right) + (1 - |x|)^n \to 1 \quad \text{over } [0,1] \text{ as } n \to \infty$$
   we have
   $$\begin{aligned} &\int_0^1 nx \sin\left(\frac{1}{nx}\right) + (1-|x|)^n \\ = \; &\underbrace{\frac{1}{n}\int_0^n x \sin\left(\frac{1}{x}\right) \, dx}_{1} + \underbrace{\frac{-(1-x)^{n+1}}{n+1}\Big|_0^1}_{0} \end{aligned}$$
2. For
   $$f_n(x) = \begin{cases} nx^{n-1} & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Note that as $n \to \infty$, we have

$$f_n \to f = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

3. If $f_n(x) \to 0$, is it possible that $\displaystyle\int_0^1 f_n(x)\,dx = 1$.

Theorem 0.13

If $f_n \rightrightarrows f$, then $\displaystyle\lim_n \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Example 0.34

We have

$$\int_0^{\frac{1}{2}} \sum_{n=1}^{\infty} x^n\,dx = \sum_{n=1}^{\infty} \int_0^{\frac{1}{2}} x^n\,dx$$

since

$$\sum_{n=1}^{N} x^n = \sum_{n=1}^{\infty} \qquad \text{over } [0, 1/2]$$

Theorem 0.14: Monotone Convergence Theorem

If $f_n \to f$ over $[a, b]$, and $f_1 \le f_2 \le \dots$ then

$$\begin{aligned} \lim_n \int_a^b f_n(x)\,dx &= \int_a^b \lim_n f_n(x)\,dx \\ &= \int_a^b f_n(x)\,dx \end{aligned}$$

Theorem 0.15: Dini

If f_n and f continuous and $f_n(x) \nearrow f(x)$ for all $x \in [a, b]$, so

$$f_n \rightrightarrows f$$

Proof: $\forall\, \epsilon > 0$, we want N such that if $n \ge N$, so $0 < f(x) - f_n(x) < \epsilon$ for any $x \in [a, b]$. Define

$$U_n := \Big\{ x \in [a, b] : f(x) - f_n(x), \epsilon \Big\}$$

Claim: $U_N = U_{N+1} = \cdot = [a, b]$ for large N.
Proof of the claim:
SFAC $\forall\, m$, $\exists\, x_m \in [a, b] \backslash U_m$. However, $x_m \in [a, b] \;\Rightarrow\; x_m \to x \in [a, b]$, and since $\displaystyle\lim_{n\to\infty} \bigcup_n U_n = [a, b]$, so $x \in U_i$ for some $i \in \mathbb{N}$.

$$\Rightarrow \quad f(x) - f_i(x) < \epsilon$$

However, by what we supposed, we have $\epsilon, f(x_m) - f_m(x_m) \leq f(x_m) - f_i(x_m) \leq \epsilon$, which is a contradiction. □

Example 0.35

Considet the previous example, another justification would be

$$\sum_{n=0}^{N} x^n \nearrow \sum_{n=0}^{\infty} x^n \quad \text{over } [0, 1/2]$$

Theorem 0.16: Dominated Convergence Theorem

For f_n and f continuous and $f_n \to f$ over $[a, b]$, and $|f_n| \leq M \ll \infty$, then

$$\lim_n \int_a^b f_n(x)\, dx = \int_a^b \lim_n f_n(x)\, dx$$

TUT - Mon - Feb 26 - 2024

Weierstrass M-Test

Theorem 0.17: Weierstrass M-Test

Let $A \subseteq \mathbb{R}$, and $f_k : A \to \mathbb{R}$. Suppose $\exists\, M_k \geq 0$ such that $|f_k(x)| \leq M_k$ for all $x \in A$, and $\sum_{k=1}^{\infty} M_k \ll \infty$.

Show that $\sum_{k=1}^{\infty} f_k$ converges uniformly and absolutely on A.

Proof: Let $S_m(x) = \sum_{k=1}^{m} f_k(x)$. Let $\epsilon > 0$, becuase we know that $\sum_{k=1}^{\infty} M_k \ll \infty$, choose $N > 0$ so that $m > n > N$, then $\sum_{k=n+1}^{m} M_k < \epsilon$. THen if $x \in A$, $m > n > N$, we will have

$$|S_m(x) - S_n(x)| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} M_k < \epsilon$$

Thus for each $x \in A$, $\{S_n(x)\}_n$ is Cauchy. Then we let $S(x) = \lim_n S_n(x) = \sum_{k=1}^{\infty} f_k(x)$. If $n > N, x \in A$, then

$$\begin{aligned} |S(x) - S_n(x)| &= \left| \lim_m S_m(x) - S_n(x) \right| \\ &= \lim_m |S_m(x) - S_n(x)| \\ &\leq \lim_m \epsilon \\ &= \epsilon \end{aligned}$$

□

Example 0.36

Show that $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}$ converges uniformly on $\mathbb{R}$ and compute $\displaystyle\int_0^{\pi/4} \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{1+n^2}$.

Example 0.37

Show that $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly on any closed interval $[a, b]$.

Lec 19 - Mon - Feb 26 - 2024

We talked about Midterm

Lec 20 - Wed - Feb 28 - 2024

We talked about Midterm solutions

Lec 21 - Fri - Mar 1 - 2024

Remember from last lecture, we have that if

$$f_n(x) \to f(x) \qquad f_n'(x) \to g(x)$$

and $f_n'(x)$ is integrable, then we have

$$\begin{aligned} f_n(x) &= \int_{x_0}^{x} f_n'(x)\,dx + f_n(x_0) \\ \Rightarrow \quad f(x) &= \lim_n \int_{x_0}^{x} f_n'(x)\,dx + f(x_0) \\ \Rightarrow \quad f(x) &= \int_{x_0}^{x} g(x)\,dx + f(x_0) \end{aligned}$$

the last arrow holds if, for exmaple, $f_n'(x) \rightrightarrows g(x)$. Therefore, we find that if $f_n'(x)$ converges to $g(x)$ uniformly, then

$$\frac{df}{dx} = g$$

Example 0.38

Justify that

$$xe^{-x}e^{-xh} \rightrightarrows xe^{-x}$$

as $h \to 0$ over $[0, 1]$.

Proof: #1 We first use contradiction to prove that for all $\epsilon > 0$, $\exists\, \delta > 0$ so that if $|h| < \delta$, then

$$|xe^{-x}e^{-xh} - xe^{-x}| < \epsilon$$

$\forall\, x \in [0,1]$. SFAC that $\exists\, \epsilon_0 > 0$ such that $\forall\, \frac{1}{i}$, $\exists\, |h_i| < \frac{1}{i}$, $x_i \in [0,1]$, but

$$|xe^{-x}e^{-xh} - xe^{-x}| > \epsilon_0$$

However, $x_i \to x_\infty$ and $h_i \to 0$, which implies that

$$|xe^{-x}e^{-xh} - xe^{-x}| = |xe^{-x} - xe^{-x}| = 0 > \epsilon_0$$

which is a contradiction. $\square$

Proof: #2 It STP that

$$\left|\left(xe^{-x}(1 - e^{-xh})\right)'\right| \leq C \ll \infty$$

Let $f_h(x) := xe^{-x}(1 - e^{-xh})$, so it STP $f_h \rightrightarrows f_0$ and $|f_h'| \leq C$. The key in this method is that

$$f_h(x_h) \to f_0(x)$$

if $x_h \to x$ and $h \to 0$. Then we notice

$$f_h(x_h) - f_0(x) = \underbrace{f_h(x_h) - f_h(x)}_{f_h'(\xi)(x_h - x)} + f_h(x) - f_0(x)$$

$\square$

Result 0.11

If $f_n \rightrightarrows f$ over $[a,b]$, then

(1) f_n continuous implies f is continuous

(2) $\lim_n \int f_n = \int \lim_n f_n$

(3) $f_n' \rightrightarrows g$ implies that f exists and $f' = g$, and

$$\lim_n \frac{d}{dx} f_n = \frac{d}{dx} \lim_n f_n$$

Power Series

Notice

$$\begin{aligned} f_N &= \sum_{n=1}^{N} a_n (x-c)^n \\ \Rightarrow \quad f_N' &= \sum_{n=1}^{N} n a_n (x-c)^{n-1} \end{aligned}$$

We can take teh derivative inside since we are summing up a finite sum. Thus if

$$\overline{\lim_{n\to\infty}} |a_n(x-c)^n|^{\frac{1}{n}} \le L < 1$$

then the series converges, where

$$\overline{\lim_{n\to\infty}} |a_n(x-c)^n|^{\frac{1}{n}} = \lim_{k\to\infty}\left(\sup_{n>k} |a_n|^{\frac{1}{n}}(x-c)\right)$$

Therefore it suffices to have (radius of convergence)

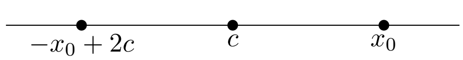
$$|x-c| < R := \frac{1}{\overline{\lim}_{n\to\infty}|a_n|^{\frac{1}{n}}}$$

Notice that R' for f_N' is

$$\begin{aligned} R' &= \frac{1}{\overline{\lim}_{n\to\infty}|a_n|^{\frac{1}{n}} n^{\frac{1}{n}}} \\ &= \frac{1}{\overline{\lim}_{n\to\infty}|a_n|^{\frac{1}{n}}} \\ &= R \end{aligned}$$

Suppose $\sum_{n=1}^{\infty} |a_n(x_0-c)^n|$ converge, then we would have

1. $$\sum_{n=1}^{N} a_n(x_0-c)^n \quad \rightrightarrows \quad \sum_{n=1}^{\infty} a_n(x_0-c)^n := S(x)$$



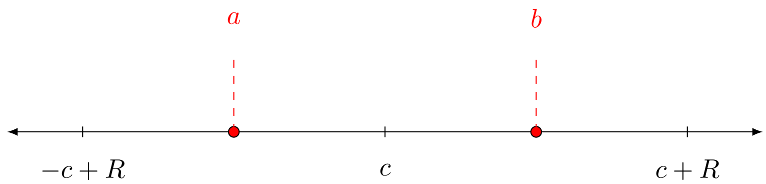
over $[-x_0+2c, x_0]$

2. $$\sum_{n=1}^{N} n\cdot a_n(x_0-c)^{n-1} \quad \rightrightarrows \quad \sum_{n=1}^{\infty} n\cdot a_n(x_0-c)^{n-1} := S'(x)$$

   over any proper subinterval $[a,b] \subseteq [-x_0+2c, x_0]$ where $a \neq -x_0+2c$ and $b \neq x_0$.

Result 0.12

Over the interval $|x-c|<R$, we can do



1. $$\int_a^b \sum_{n=0}^{\infty} a_n (x-c)^n \, dx = \sum_{n=0}^{\infty} a_n \int_a^b (x-c)^n \, dx$$

2. $$\frac{d^k}{dx^k} \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} \frac{d^k}{dx^k} a_n (x-c)^n$$

Lec 22 - Mon - Mar 4 - 2024

Goal 0.16

Our goal is to justify that
$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

Method 1 : Lower Sum

$$\begin{aligned}
1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N-1} + \frac{1}{2N} &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2N}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2N}\right) \\
&= \left(1 + \frac{1}{2} + \cdots + \frac{1}{2N}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right) \\
&= \frac{1}{N+1} + \cdots + \frac{1}{2N} \\
&= \frac{1}{N}\frac{1}{1+\frac{1}{N}} + \cdots + \frac{1}{N}\frac{1}{1+\frac{N}{N}}
\end{aligned}$$

which is essentially the lower sum of the function $f(x) := \frac{1}{x+1}$ over the interval $[0,1]$: notice that the above expression is also equal to

$$\begin{aligned}
\sum_{k=1}^{N} \frac{1}{N+k} &= \sum_{k=1}^{N} \frac{1}{N}\frac{1}{1+\frac{k}{N}} \\
&= \sum_{k=1}^{N} \Delta x_k f(x_k)
\end{aligned}$$

Therefore, as $N \to \infty$, we can find that it approaches $\ln 2$ (We can take the integral of the function from 0 to 1).

Method 2 : Euler-Mascheroni γ Constant

It is easy to notice that

$$\lim_n \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$$

exists, since we know that

$$\begin{aligned} a_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \quad \searrow \\ b_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1) \quad \nearrow \end{aligned}$$

and easy to find that

$$\lim_{n\to\infty} a_n - b_n = \ln\left(\frac{n+1}{n}\right) \to 0$$

Therefore, as a result,

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2N-1} + \frac{1}{2N} &= \left(1 + \cdots + \frac{1}{2N}\right) - 2 \cdot \frac{1}{2}\left(1 + \cdots + \frac{1}{N}\right) \\ &= \left(1 + \cdots + \frac{1}{2N}\right) - \ln 2N - \left(\left(1 + \cdots + \frac{1}{N}\right) - \ln N\right) + \ln 2 \\ &= \ln 2 \end{aligned}$$

Method 3 : Taylor Expansion

We know that

$$f(x) = \ln(1+x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{N-1}x^N}{N} + \frac{f^{(N+1)}(\xi_N)}{(N+1)!}x^{N+1}$$

and

$$(\ln(1+x))^N = (-1)^{N-1} \cdot \frac{(N-1)!}{(1+x)^N}$$

Therefore, we have

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{N-1}}{N} + (-1)^N \frac{N!}{(1+\xi_N)^N (N+1)!}$$

Notice that $(-1)^N \dfrac{N!}{(1+\xi_N)^N (N+1)!} = 0$ as $N \to \infty$. Thus completing the proof.

Method 4:

We know that

$$\begin{aligned} \frac{1}{x+1} &= 1 - x + x^2 - x^3 + x^4 - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^n x^n \end{aligned}$$

Therefore, we can know that

$$
\begin{aligned}
\int_0^t \frac{1}{x+1}\,dx &= \int_0^t \sum_{n=1}^{\infty}(-1)^n x^n\,dx \\
\Rightarrow \quad \ln(1+t) &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots
\end{aligned}
$$

as long as $|t| < 1$. Note that when $t = 1$, LHS $= \ln 2$, and the RHS is equal to the infinite series we want. However, the things that is missing is that when $t = 1$, we cannot necessarily take the integral. Therefore we consider the following process:

$$
\begin{aligned}
\ln 2 = \lim_{t \to 1^-} \ln(1+t) &= \lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1} \\
&= \sum_{n=0}^{\infty} \lim_{t \to 1^-} \frac{(-1)^n}{n+1} t^{n+1}
\end{aligned}
$$

if we can commute the limit and the sum. Therefore it remains to show that

$$
\lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1} = \sum_{n=0}^{\infty} \lim_{t \to 1^-} \frac{(-1)^n}{n+1} t^{n+1}
$$

Lec 23 - Wed - Mar 6 - 2024

Goal 0.17

We want to show that

$$
\lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^{n+1} = \sum_{n=0}^{\infty} \lim_{t \to 1^-} \frac{(-1)^n}{n+1} t^{n+1}
$$

Equivalently, we would like to show that

$$
\lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (t^{n+1} - 1) = 0
$$

Proof: Want to show that $\forall\, \epsilon > 0$, $\exists\, \delta$ so that if $1 - \delta < t < 1$, then

$$
\lim_{t \to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (t^{n+1} - 1) < \epsilon
$$

1. Step 1:
   Choose N large so that
   $$
   \left| \sum_{n=N}^{\infty} \frac{(-1)^n}{n+1} (t^{n+1} - 1) \right| < \frac{\epsilon}{2}
   $$
   for all $0.9 < t < 1$.
   Recall Abel's test, we define
   $$
   S_m := \sum_{n=1}^{m} \frac{(-1)^n}{n+1}
   $$

hence

$$\begin{aligned}
\sum_{n=N}^{\infty} \frac{(-1)^n}{n+1}(t^{n+1}-1) &= \sum_{n=N}^{\infty}(S_n - S_{n-1})b_n \\
&= \sum_{n=N}^{\infty} S_n b_n - \sum_{n=N}^{\infty} S_{n-1} b_n \\
&= \sum_{n=N}^{\infty} S_n b_n - \sum_{n=N-1}^{\infty} S_n b_{n+1} \\
&= \underbrace{-S_{n-1}b_n}_{\to C\cdot(t^{N+1}-1)} + \sum_{n=N}^{\infty} S_n \underbrace{(b_n - b_{n+1})}_{\to -C\cdot(t^{N+1}-1)}
\end{aligned}$$

2. Step 2:
   For fixed value of N, we want to find δ' small such that if $1-\delta' < t < 1$, then

$$\left|\sum_{n=0}^{N} \frac{(-1)^n}{n+1}(t^{n+1}-1)\right| < \frac{\epsilon}{2}$$

   Remark: Note that this is essentially the proof for continuity.

Therefore, we can simply choose

$$\delta = \min\{0.1,\ \delta'\}$$

thus completing the proof. □

Result 0.13

In general, if $\sum_{n=1}^{\infty} a_n x^n$ converges (uniformly) over $|x| < R$ and $\sum_{n=1}^{\infty} a_n R^n$ also converges, athen we have

$$\begin{aligned}
\int_0^R f(x)\,dx &= \int_0^R \sum_{n=0}^{\infty} a_n x^n\,dx \\
&= \sum_{n=0}^{\infty} a_n \int_0^R x^n\,dx
\end{aligned}$$

Goal 0.18

Our goal is

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

Notice that we have

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{x}\,dx &= \int_0^1 \frac{x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots}{x}\,dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int_0^1 \frac{x^n}{n+1}\,dx \\
&= 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \\
&= \left(1+\frac{1}{2^2}+\frac{1}{3^2}\right)-2\left(\frac{1}{2^2}+\frac{1}{4^2}+\cdots\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \\
&= \frac{\pi^2}{12}
\end{aligned}$$

Therefore it remains to show the following

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Proof: We notice that

$$\sin x \le x \le \tan x \qquad \text{for } x \in \left[0, \frac{\pi}{2}\right]$$

Thus we have

$$\begin{aligned}
\frac{1}{(\sin x)^2} \;&\ge \frac{1}{x^2} \;\ge\; \frac{(\cos x)^2}{(\sin x)^2} \\
\Rightarrow\quad \frac{1}{(\sin x)^2}-1 \;&\le \frac{1}{x^2} \;\le\; \frac{1}{(\sin x)^2}
\end{aligned}$$

Define $x_k := \frac{k}{2}\cdot\frac{\pi}{2}$, and substitute it back to the inequality we obtained,

$$\frac{1}{(\sin x_k)^2}-1 \le \frac{1}{x_k^2} \le \frac{1}{(\sin x_k)^2}$$

where we also define $\sum_{k=1}^{2^n-1}\frac{1}{(\sin x_k)^2}=S_n$. As a result, we can easily find that

$$S_n-(2^n-1) \le \sum_{k=1}^{2^n-1}\frac{1}{x_k^2} \le S_n$$

Remark: We need to think about this in the sense of geometry, refer to $\mathfrak{Xuemiao}$'s notes for details.

□

Goal 0.19: Stirling's Approximation

Our goal is to justify the Stirling's Approximation, that is

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}$$

for some $\theta \in (0,1)$.

We first introduce a_n to be

$$a_n = \frac{n!\, e^n}{n^{n+\frac{1}{2}}}$$

Therefore it suffices to prove that

$$\lim_{n\to\infty} a_n = \sqrt{2\pi}\cdot e^{\frac{\theta_n}{12n}}$$

Notice that

$$\begin{aligned}\frac{a_n}{a_{n+1}} = \frac{\frac{n!\, e^n}{n^{n+\frac{1}{2}}}}{\frac{(n+1)!\, e^{n+1}}{(n+1)^{n+1+\frac{1}{2}}}} &= \frac{\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}}{e} \\ &= \frac{e^{\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right)}}{e}\end{aligned}$$

We can discover that

$$\begin{aligned}\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) &= \left(n+\frac{1}{2}\right)\cdot 2\cdot\frac{1}{2n+1}\left(1+\frac{\frac{1}{(2n+1)^2}}{3}+\cdots+\frac{\frac{1}{(2n+1)^{2k}}}{2k+1}+\cdots\right) \\ &= 1+\frac{\frac{1}{(2n+1)^2}}{3}+\cdots+\frac{\frac{1}{(2n+1)^{2k}}}{2k+1}+\cdots \\ &\le 1+\frac{1}{3}\left(\frac{1}{(2n+1)^{2k}}+\cdots+\frac{1}{(2n+1)^{2k}}+\cdots\right) \\ &= 1+\frac{1}{3}\cdot\frac{\frac{1}{(2n+1)^2}}{1-\frac{1}{(2n+1)^2}} \\ &= 1+\frac{1}{12}\cdot\frac{1}{n(n+1)}\end{aligned}$$

Therefore

$$\begin{aligned} & 1 \le \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) \le 1+\frac{1}{12}\cdot\frac{1}{n(n+1)} \\ \Rightarrow\quad & 1 \le \frac{a_n}{a_{n+1}} \le e^{\frac{1}{12n(n+1)}} \\ \Rightarrow\quad & \frac{a_n}{e^{\frac{1}{12n}}} \le \frac{a_{n+1}}{e^{\frac{1}{12(n+1)}}}\end{aligned}$$

and since $a_n \searrow$ and $\frac{a_n}{e^{\frac{1}{12n}}} \nearrow$, and

$$\frac{a_n}{e^{\frac{1}{12n}}} < a_n$$

so we know that there exists $a \in \left(\frac{a_n}{e^{\frac{1}{12n}}}, a_n\right)$ such that

$$\begin{aligned}
& \frac{a_n}{e^{\frac{1}{12n}}} \leq a \leq a_n \\
\Rightarrow \quad & a \leq a_n \leq ae^{\frac{1}{12n}} \\
\Rightarrow \quad & ae^{\frac{0}{12n}} \leq a_n \leq ae^{\frac{1}{12n}} \\
\Rightarrow \quad & a_n = ae^{\frac{\theta_n}{12n}} \qquad \text{for } \theta_n \in (0,1) \\
\Rightarrow \quad & n! = \sqrt{n}a\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}
\end{aligned}$$

Theorem 0.18: Wallis

We have

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$$

Note that

$$\begin{aligned}
\frac{(2n)!!}{(2n-1)!!} = \frac{((2n)!!)^2}{(2n)!} &= \frac{(2^n n!)^2}{(2n)!} \\
&= \frac{\left[2^n \cdot a\sqrt{n}\left(\frac{n}{e}\right)^n\right]^2}{a\sqrt{2n}\left(\frac{2n}{e}\right)^{2n}} \\
&= \frac{a}{\sqrt{2}}\sqrt{n} \\
\Rightarrow \quad \lim_{n\to\infty} \frac{1}{2n+1} \cdot \frac{a^2 n}{2} &= \frac{\pi}{2}
\end{aligned}$$

After computing we can obtain that $a = \sqrt{2}$.

Lec 25 - Mon - Mar 11 - 2024

Goal 0.20

Here we justify why

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) = 1 + \frac{1}{3}\frac{1}{(2n+1)^2} + \frac{1}{5}\frac{1}{(2n+1)^4} + \cdots$$

Proof: We have for $|x| < 1$,

$$\begin{cases} \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \\ \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \end{cases}$$
$$\Rightarrow \quad \begin{cases} -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \\ -\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \end{cases}$$
$$\Rightarrow \quad \ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^2}{3} + \frac{2x^5}{5} + \cdots$$
$$\Rightarrow \quad \frac{1}{2x} \ln\left(\frac{1+x}{1-x}\right) = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots$$

Therefore we can simply let $x = \dfrac{1}{2n+1}$, so we can have the desired outcome. □

Goal 0.21

We also want to justify Wallis Lemma.

Proof: We have

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{(2n)!!}{(2n+1)!!}$$
$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{\pi}{2} \frac{(2n-1)!!}{(2n)!!}$$

Moreover, we know that

$$\sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x$$
$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n+1} \, dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n} \, dx \leq \int_0^{\frac{\pi}{2}} \sin^{2n-1} \, dx$$
$$\Rightarrow \frac{(2n)!!}{(2n+1)!!} \leq \frac{\pi}{2} \frac{(2n-1)!!}{(2n)!!} \leq \frac{(2n-2)!!}{(2n-1)!!}$$
$$\Rightarrow \frac{2n}{2n+1} \frac{\pi}{2} \leq \frac{1}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \leq \frac{\pi}{2}$$

as desired. □

Goal 0.22

Our goal this time is to prove that

$$\begin{aligned}\frac{\sin x}{x} &= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots \\ &= \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\end{aligned}$$

Lec 26 - Wed - Mar 13 - 2024

1

Notice that the coefficient for x^3 for LHS and RHS respectively is

$$\begin{aligned}-\frac{1}{6} &= -\frac{1}{\pi^2}-\frac{1}{(2\pi)^2}-\cdots \\ \Rightarrow \quad \frac{\pi^2}{6} &= 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\end{aligned}$$

2

Notice that

$$\begin{aligned}\cos x = \frac{\sin 2x}{2\sin x} &= \frac{2x\left(1-\frac{(2x)^2}{\pi^2}\right)\left(1-\frac{(2x)^2}{(2\pi^2)}\right)\cdots}{2x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots} \\ &= \left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\cdots\end{aligned}$$

Proof: Meaning of the RHS:

$$\begin{aligned}RHS &= \lim_{N\to\infty} x\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{N^2\pi^2}\right) \\ &= \lim_{N\to\infty} e^{\ln x\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{N^2\pi^2}\right)} \\ &= \lim_{N\to\infty} e^{\sum_{n=1}^{N}\ln\left(1-\frac{x^2}{n^2\pi^2}\right)}\end{aligned}$$

Formula:

$$\begin{aligned}\sin x &= \sin\left((2n+1)\frac{x}{2n+1}\right) \\ &= \sin\left((2n+1)y\right) \\ &= \sin y\cdot P(\sin^2 y)\end{aligned}$$

where $P(t) = a_n t^n + \cdots + a_0$. Hence our goal is to show that

$$\begin{aligned}\sin((2n+1)y) = \sin x = \sin y \cdot P(\sin^2 y) \\ = (2n+1)\sin y \cdot \left(1 - \frac{\sin^2 y}{\sin^2 \frac{\pi}{2n+1}}\right) \cdots \left(1 - \frac{\sin^2 y}{\sin^2 \frac{n\pi}{2n+1}}\right)\end{aligned}$$

the leading "coefficient", $2n+1$, is obtained from moving $\sin y$ to the left and take $y = 0$. Notice that when $n \to \infty$, we can find that

$$(2n+1)\sin\frac{x}{2n+1} \to x$$

$$1 - \frac{\sin^2 \frac{x}{2n+1}}{\sin^2 \frac{\pi}{2n+1}} = \frac{\sin^2 \frac{x}{2n+1}}{\left(\frac{x}{2n+1}\right)^2} \cdot \frac{\left(\frac{x}{2n+1}\right)^2}{\left(\frac{\pi}{2n+1}\right)^2} \cdot \frac{\left(\frac{\pi}{2n+1}\right)^2}{\sin^2 \frac{\pi}{2n+1}} \to \frac{x^2}{\pi^2}$$

And the following part are obtained from

$$\begin{aligned}\sin((2n+1)y) &= \operatorname{Im}\left(e^{iy(2n+1)}\right) \\ &= \operatorname{Im}\left((\cos y + i\sin y)^{2n+1}\right)\end{aligned}$$

□

Lec 27 - Fri - Mar 15 - 2024

Continue from last lecutre, let $n \to \infty$, we would have

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots$$

However, notice that that in regard to the last term, we dont have

$$\sin^2 \frac{n\pi}{2n+1} = \left(\frac{\pi}{2}\right)^2$$

as $n \to \infty$. But we can notice a pattern:

$$\begin{array}{lllll} n & & & & \\ 0 & \sin x & & & \\ 1 & 3\sin\frac{x}{3} & 1 - \frac{\sin^2 \frac{x}{3}}{\sin^2 \frac{\pi}{3}} & & \\ 2 & 5\sin\frac{x}{5} & 1 - \frac{\sin^2 \frac{x}{5}}{\sin^2 \frac{\pi}{5}} & 1 - \frac{\sin^2 \frac{x}{5}}{\sin^2 \frac{2\pi}{5}} & \\ \vdots & \vdots & \vdots & \vdots & \\ \infty & & & & \to \sin x \end{array}$$

Notice how the tail does not matter, here is a better proof:

We want to show that

$$\lim_{n\to\infty}(2n+1)\sin\frac{x}{2n+1}\cdot\left(1-\frac{\sin^2 y}{\sin^2\frac{\pi}{2n+1}}\right)\cdots\left(1-\frac{\sin^2 y}{\sin^2\frac{n\pi}{2n+1}}\right)$$
$$= x\left(1-\frac{x^2}{\pi^2}\right)\cdots$$

Proof:

1. Step 1:
   We first choose N large so that
   $$\prod_{k=N}^{n}\left(1-\frac{\sin^2 y}{\sin^2\frac{k\pi}{2n+1}}\right)s \ > 1-\epsilon$$
   for any $n \geq n_0$
2. Step 2:
   For N fixed from step 1, there exists n_1 large so that if $n > n_1$,
   $$\left|(2n+1)\sin\frac{x}{2n+1}\prod_{k=1}^{N-1}\left(1-\frac{\sin^2 y}{\sin^2\frac{k\pi}{2n+1}}\right)-x\left(1-\frac{x^2}{\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\right|<\epsilon$$

Combining step 1 and step 2, we take $n \geq \max\{n_0, n_1\}$, so the following follows naturally. □

Lec 28 - Mon - Mar 17 - 2024

Goal 0.23

Define $f_n : [0,\infty) \to \mathbb{R}$ by
$$f_n(x) = nxe^{-nx}$$
Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0,\infty)$.

Using L'Hopital's Rule, we can obtain that

$$\lim_{t\to\infty}\frac{tx}{(e^x)^t} = \lim_{t\to\infty}\frac{1}{(e^x)^t} = 0$$

and at $x = 0$, we also have $nxe^{-nx} = 0 \to 0$, hence we can find that

$$nxe^{-nx} \to 0 \qquad \text{over} \ \ [0,\infty)$$

Notice that the maximum of the function is at $x = \frac{1}{n}$, where the function is evaluated to be e^{-1}, which does not converges to 0, thus we can conclude that the function is not uniformly convergent to 0.
Exercise: Show:
$$f_n \rightrightarrows f \quad \Rightarrow \quad f_n(x_n) \to f(x) \ \text{ as } \ x_n \to x$$

we simply have

$$\begin{aligned} f_n(x_n) - f(x) &\overset{A}{=} f_n(x_n) - f_n(x) + f_n(x) - f(x) \\ &\overset{B}{=} \underbrace{f_n(x_n) - f(x_n)}_{\text{uni conv}} + \underbrace{f(x_n) - f(x)}_{\text{continuity}} \end{aligned}$$

Goal 0.24

Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{x}{1 + nx^2}$$

Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0, \infty)$.

Notice that function converges to 0, because we can rewrite the function as

$$f_n(x) = \frac{\frac{x}{n}}{\frac{1}{n} + x^2}$$

so the pointwise limit of the function is 0. Easy to notice that the function converges to 0 pointwisely over $[0, \infty)$, so we can only consider the interval $[0, 1]$. Recall that $a^2 + b^2 > ab$, so we have

$$\frac{x}{1 + nx^2} \leq \frac{x}{2\sqrt{nx^2}} = \frac{1}{2\sqrt{n}}$$

hence the function uniformly converges to 0.

Goal 0.25

Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{x + n}{x + 4n}$$

Find the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ and determine whether $f_n \to f$ uniformly on $[0, \infty)$.

We can find that the function converges to $\frac{1}{4}$ since we can rewrite the function as

$$f_n(x) = \frac{x + n}{x + 4n} = \frac{\frac{x}{n} + 1}{\frac{x}{n} + 4} \to \frac{1}{4}$$

We first prove that the function is not uniformly convergent over $[0, \infty)$, we can simply take $x = n$ for all values of n.

Remark: Using the definition to prove the above statement.

Goal 0.26

Find $\displaystyle\int_0^1 \lim_{n\to\infty} nx(1 - x^2)^n \, dx$ and $\displaystyle\lim_{n\to\infty} \int_0^1 nx(1 - x^2)^n \, dx$.

We can find that the first integral evaluate to be 0 since the limit goes to 0. For the second integral, we have

$$\begin{aligned}\int_0^1 nx(1-x^2)^n\,dx &= -\frac{1}{2}\int_0^1 n(1-x^2)^n\,d(1-x^2)\\ &= -\frac{1}{2}\cdot\frac{n}{n+1}(1-x^2)^{n+1}\Big|_0^1 \to \frac{1}{2}\end{aligned}$$

thus we can conclude that the function is not uniform convergent, because we have different solution when commuting the integral and the limit.

Remark: The bad point in the function is when x approaches to 0, specifically, at $x = \frac{1}{n}$.

Lec 29 - Wed - Mar 20 - 2024

Goal 0.27

Find

$$\lim_{n\to\infty}\int_1^4 \frac{\arctan(nx)}{x}\,dx = \int_1^4 \lim_{n\to\infty}\frac{\arctan(nx)}{x}\,dx$$

Easy to find that the RHS is simply $\frac{\pi}{2}\ln 4$. For the LHS, we can use the Mean Value Theorem to have

$$\lim_{n\to\infty}\int_1^4 \frac{\arctan(nx)}{x}\,dx = \arctan(x\xi)\int_1^4 \frac{1}{x}\,dx$$

On the other hand, we can try to prove that

$$\frac{\arctan(nx)}{x} \rightrightarrows \frac{\pi}{2}\cdot\frac{1}{x} \qquad \text{over } [1,4]$$

We can use the dominated convergence theorem, using the fact that

$$\left|\frac{\arctan(nx)}{x}\right| \le C \lll \infty$$

Alternatively, we can also use Dini with the fact that

$$\frac{\arctan(nx)}{x} \nearrow \frac{\pi}{2}\cdot\frac{1}{x}$$

Goal 0.28

Show that

$$\sum_{n=1}^{N}\frac{\cos(2^n x)}{1+n^2} \rightrightarrows \sum_{n=1}^{\infty}\frac{\cos(2^n x)}{1+n^2}$$

Our goal is to show that for all ϵ, there exists N_0 so that if $N_1 > N_2 > N_0$, then

$$\left|\sum_{n=1}^{N_1}\frac{\cos(2^n x)}{1+n^2} - \sum_{n=1}^{N_2}\frac{\cos(2^n x)}{1+n^2}\right| \qquad \text{for any } x\in\mathbb{R}$$

Notice that the LHS is less than or equal to

$$\sum_{n=N_1}^{N_2} \frac{\cos(2^n x)}{1+n^2} \leq \sum_{n=N_1}^{N_2} \frac{1}{1+n^2} \leq \int_{N_1}^{\infty} \frac{1}{1+x^2}\, dx$$

Goal 0.29

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$ converges uniformly on any closed interval $[a, b]$.

We have $\sin\left(\frac{x}{n^2}\right) \approx \frac{x}{n^2}$

Goal 0.30

Is it true that: If (f_n) and (g_n) converge uniformly on E then $(f_n g_n)$ converge uniformly on E.

Proof: Wrong, consider $f_n = x + \frac{1}{n}$ and $g_n = x + \frac{1}{n}$, notice that $f_n g_n = x^2 + \frac{2x}{n} + \frac{1}{n^2}$, which is not uniform convergent because $\frac{2x}{n} \not\to 0$. □

Goal 0.31

] Show that if (f_n) and (g_n) converge uniformly on E and f and g are bounded on E then $(f_n g_n)$ converges uniformly on E.

Proof: We have

$$\begin{aligned} |f_n g_n - fg| &= |f_n(g_n - g) + (f_n - f)g| \\ &\leq |g_n - g||f| + |f_n - f||g| \end{aligned}$$

wwwww □

Lec 30 - Fri - Mar 22 - 2024

Goal 0.32

We want to identify whether

$$\sum_{n=1}^{N} \frac{(-1)^n x^n}{n + \ln n} \rightrightarrows \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n + \ln n} \qquad \text{over } [0,1]$$

Proof: We define

$$S_n := \sum_{k=1}^{n} \frac{(-1)^n}{k + \ln k}$$

We would like to show that LHS is a uniform Cauchy sequence

$$
\begin{aligned}
\left|\sum_{n=1}^{N_1}\cdots-\sum_{n=1}^{N_2}\cdots\right| &= \left|\sum_{N_1}^{N_2}\cdots\right| \\
&= \left|\sum_{N_1}^{N_2}(S_n-S_{n-1})x^n\right| \\
&= \left|\sum_{N_1}^{N_2}S_nx^n-\sum_{n=N_1}^{N_2}S_{n-1}x^n\right| \\
&= \left|\sum_{n=N_1}^{N_2-1}S_n(x^n-x^{n+1})+S_{N_2}x^{N_2}-S_{N_1-1}x^{N_1}\right| \\
&\le \left|\sum_{n=N_1}^{N_2-1}S_n(x^n-x^{n+1})\right|+\left|S_{N_2}x^{N_2}-S_{N_1-1}x^{N_1}\right| \\
&\le \sum_{n=N_1}^{N_2-1}\left|S_n\right|(x^n-x^{n+1})+\left|S_{N_2}x^{N_2}-S_{N_1-1}x^{N_1}\right|
\end{aligned}
$$

where we notice that the left sum is uniform, but it is hard to deal with the difference on the right. Alternatively, we could argue that the numerator is bounded, thus we could have

$$
S_n := \sum_{k=1}^{n}(-1)^kx^k = \frac{(-x)-(-x)^{n+1}}{1+x}
$$

then

$$
\left|\sum_{N_1}^{N_2}\frac{1}{n+\ln n}(S_n-S_{n-1})\right| = \left|\frac{1}{N_1+\ln N_1}S_{N_1}-\frac{1}{N_2+\ln N_2}S_{N_2-1}+\sum_{n=N_2}^{N_1}\left(\frac{1}{n+\ln n}-\frac{1}{n+1}\ln n+1\right)S_n\right|
$$

which is uniformly convergent. $\square$

Goal 0.33

Find the Taylor series centred at 0, and its interval of convergence, for $f(x)=\dfrac{x}{x^2-6x+8}$.

Proof: Notice that we have $f(x)=\dfrac{x}{(x-2)(x-4)}=\dfrac{2}{x-4}-\dfrac{1}{x-2}=-\dfrac{2}{4}\dfrac{1}{1-\frac{x}{4}}+\dfrac{1}{2}\dfrac{1}{1-\frac{x}{2}}$:

$$
\begin{aligned}
-\frac{2}{4}\frac{1}{1-\frac{x}{4}}+\frac{1}{2}\frac{1}{1-\frac{x}{2}} &= -\frac{2}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}\left(-\frac{1}{2}\right)^n-\frac{1}{2}\left(-\frac{1}{4}\right)^n\right)x^n
\end{aligned}
$$

Exercise: Check the radius of convergence.

wwwww $\square$

Goal 0.34

Find the Taylor series centred at $\frac{\pi}{4}$, and its interval of convergence, for $f(x) = \sin x \cos x$

Proof: We have

$$\begin{aligned} f(x) &= \frac{1}{2}\sin(2x) \\ &= \frac{1}{2}\sin\left(2\left(x - \frac{\pi}{4}\right) + \frac{\pi}{2}\right) \\ &= \frac{1}{2}\cos\left(2\left(x - \frac{\pi}{4}\right)\right) \end{aligned}$$

Recall that the taylor's expansion for cos at $x = 0$ is

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$$

thus

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n \left(2\left(x - \frac{\pi}{4}\right)\right)}{(2n)!} \qquad \text{over } \mathbb{R}$$

□

Goal 0.35

Let $0 < a < b$. Note that $\mathbb{Q} \cap [a, b]$ is countable, say $\mathbb{Q} \cap [a, b] = \{q_1, q_2, q_3, \ldots\}$. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} q_n x^n$.

Proof: To find the radius of convergence, we may find

$$\limsup_{n\to\infty} \left(|q_n x^n|\right)^{\frac{1}{n}} < 1$$

from which we can obtain that the radius of convergence is $R = 1$. □

Goal 0.36

Find the 4^{th} Taylor polynomial centred at 0 for $f(x) = \frac{\ln(1+x)}{e^{2x}}$.

Proof: Long division □

Goal 0.37

Let $f(x) = x^3 + x + 1$. Note that f is increasing with $f(0) = 1$, and let $g(x) = f^{-1}(x)$. Find the 6th Taylor polynomial centred at 1 for the inverse function $g(x)$.

Proof: $g(f(x)) = x \quad \Rightarrow \quad g(x^3 + x + 1) = x$. □

Lec 31 - Mon - Mar 25 - 2024

Goal 0.38

Let $f(x) = (8 + x^3)^{2/3}$. Find $f^{(9)}(0)$, the 9th derivative of f at 0.

Proof: Recall the Binomial Expansion, we have

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \text{for } |x| < 1$$

Hence,

$$\begin{aligned}(8+x^3)^{2/3} &= 8^{2/3}\left(1+\left(\frac{x}{2}\right)^3\right)^{2/3} \\ &= 8^{2/3}\left(1+\binom{\frac{2}{3}}{1}\left(\frac{x}{2}\right)^3+\cdots+\binom{\frac{2}{3}}{3}\left(\left(\frac{x}{2}\right)^3\right)^3+\cdots\right)\end{aligned}$$

Hence the nineth derivative is given by the term with power of 9, where the answer is

$$8^{2/3}\binom{\frac{2}{3}}{3}\left(\frac{1}{2}\right)^{3\cdot 3}\underbrace{\frac{d^9x^9}{dx^9}}_{=9!}$$

□

Goal 0.39

Evaluate $\lim_{x\to 0} \frac{xe^{x^2} - \sin x}{x - \tan^{-1} x}$.

Proof: We prove this using Taylor Expansion

$$xe^{x^2} = x\left(1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots\right) \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{2}$$

$$x = x \tag{3}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{4}$$

□

Goal 0.40

Solve $y'' + y' - 2y = x + 1$ with $y(0) = 1$ and $y'(0) = 2$.

Proof: The formal solution is to write out y as a polynomial of x and compate the correpsonding coefficients. Suppose y_1 is a solution and y_0 solves $y_0'' + y_0' - 2y_0 = 0$, thus we have

$$\begin{aligned} y_1'' + y_1' - 2y_1 &= x + 1 \\ y_0'' + y_0' - 2y_0 &= 0 \\ \Rightarrow \quad (y_1 + y_0)'' + (y_1 + y_0)' - 2(y_1 + y_0) &= x + 1 \end{aligned}$$

Hence we want to find the special solution, y_1, that can be used to generate all other solutions. Notice that $y = -\frac{x}{2} + C$ is one of them. Solving for C we find that $C = -\frac{3}{4}$. Then, we would like to find a solution to $y'' + y' - 2y = 0$. Notice that the solution would be $e = e^x$ and e^{-2x}, thus completing the solution. □

Tut - Mon - Mar 27 - 2024

Goal 0.41

Our goal is to evaluate $\sqrt[5]{e}$ so that the error is at most $\frac{1}{1000}$.

Lec 32 - Wed - Mar 25 - 2024

Consider the example

$$\ln(2) = \ln(1 + 1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

The first method was that we know

$$\begin{aligned} \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-x)^n \qquad |x| < 1 \\ \text{integration} \quad \Rightarrow \quad \ln(1 + x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \end{aligned}$$

Recall that we take the limit of x approaching 1 from the left side, and we commute the limit and the sum. However, the commutativity process is the subtle step that needs to be reviewed carefully, since it does not work for all cases. It is important to note that we don't have the following comparison:

$$\left| \sum_{n=N}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \right| \leq \left| \sum_{n=N}^{\infty} \frac{(-1)^n}{n+1} \right|$$

But how can we make it right? We know that the right hand side converges, which is due to the reason that it is alternating. It gives us a limit, which can be justified by the Dirichlet Test. (Or we can think about how we can group it in pairs in two different ways, which will then show that it is bounded between two values, and it is monotone in either ways, so it definitely converges). Noticebly, the left hand side is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } x \in [0,1]$$

where we can do the same thing, that is, group the sum in pairs. If we group the first and the second, the third and the fourth ..., we can find that it turns out to be

$$\left| \sum_{n=N}^{\infty} \frac{x^{2n+1}}{2n+1} - \frac{x^{2n+2}}{2n+2} \right| \leq \left| \sum_{n=N}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+2} \right) \right|$$

but how is the inequality obvious? We notice that the function $x^{2n} - x^{2n+1} \geq 0$, which is derivative of each term on the left hand side.

Goal 0.42

Let $a_1 = \frac{7}{2}$ and for $n \geq 1$ let $a_{n+1} = \frac{6}{2-a_n}$. Determine whether (a_n) converges, and if so find the limit.

Proof: We notice that

$$a_{n+1} - a_n = \frac{6}{5-a_n} - \frac{6}{5-a_{n-1}} = \frac{6(a_n - a_{n-1})}{(5-a_n)(5-a_{n-1})}$$

Some induction is required to prove that the denominator is positive, and then we will find $x = \dfrac{6}{5-x}$. □

Goal 0.43

Let $(x_k)_{k \geq 0}$ be a sequence in $\mathbb{R}$ with $|x_k - x_{k-1}| \leq \frac{1}{k^2}$ for all $k \geq 1$. Show that (x_k) converges in $\mathbb{R}$.

Proof: We have

$$\begin{aligned} |x_{N'} - x_N| &= |x_{N'} - x_{N'-1} + x_{N'-1} - x_{N'-2} + \cdots + x_{n+1} - x_N| \\ &\leq \frac{1}{(N')^2} + \frac{1}{(N'-1)^2} + \cdots + \frac{1}{(N+1)^2} \\ &\leq \int_N^{\infty} \frac{1}{x^2}\, dx \end{aligned}$$

□

Goal 0.44

$\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}}{2n^2+1}$ convergent or divergent?

Proof: Integral test. □

Goal 0.45

$\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$ convergent or divergent?

Proof: Divergent. $\square$

Goal 0.46

$\sum_{n=1}^{\infty}\frac{n!n^n}{(2n)!}$ convergent or divergent?

Proof: Ratio test or Stirling's Approximation. $\square$

Goal 0.47

$\sum_{n=1}^{\infty}\left(n\sin^{-1}\left(\frac{1}{n}\right)-1\right)$ convergent or divergent?

Proof: Taylor expansion of inverse of sin. $\square$

Goal 0.48

Compute $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$

Proof: We can compute

$$\sum_{n=1}^{\infty}\frac{nx^{n-1}}{(n+1)!}=\left(\sum_{n=1}^{\infty}\frac{x^n}{(n+1)!}\right)'$$

Alternatively, we have $n=n+1-1$. $\square$

Lec 33 - Mon - Apr 1 - 2024

Goal 0.49

Find the sum of each of the following series, if the sum exists

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$$

Proof: We can first find that the sum is convergent because 2^n grows a lot faster than n^2. To compute, we compute

$$\begin{aligned}\sum_{n=1}^{\infty} n^2 x^n &= x \sum_{n=1}^{\infty} n^2 x^{n-1} \\ &= x \left(\sum_{n=1}^{\infty} n^2 x^n \right)' \\ &= x \left(x \sum_{n=1}^{\infty} n^2 x^{n-1} \right) \\ &= x \left(x \left(\sum_{n=1}^{\infty} n^2 x^n \right)' \right)'\end{aligned}$$

which is a geometric sequence. □

Goal 0.50

Find the sum of each of the following series, if the sum exists

$$\sum_{n=2}^{\infty} \frac{1}{a_{n-1}a_{n+1}}$$

where $\{a_n\}$ is the Fibonacci sequence.

Proof: We have

$$\begin{aligned}\frac{1}{a_{n-1}a_{n+1}} &= \frac{1}{a_{n-1}(a_n + a_{n-1})} \\ &= \frac{a_n + a_{n-1} - a_{n-1}}{a_n a_{n-1}(a_n + a_{n-1})} \\ &= \frac{1}{a_n a_{n-1}} - \frac{1}{a_n(a_n + a_{n-1})} \\ &= \frac{1}{a_n a_{n-1}} - \frac{1}{a_n a_{n+1}}\end{aligned}$$

which is in the form of $b_n + b_{n+1}$ □

Goal 0.51

Find the sum of each of the following series, if the sum exists

$$\sum_{n=3}^{\infty} \frac{2}{n^2 - 4}$$

Proof: Similar as above. □

Goal 0.52

Find the sum of each of the following series, if the sum exists

$$\sum_{n=-1}^{\infty} e^{-(n\ln 2)/2}$$

Proof: We have $e^{-(n\ln 2)/2} = e^{\ln 2^{-n/2}}$ □

Goal 0.53

Find the sum of each of the following series, if the sum exists

$$\sum_{n=2}^{\infty} \frac{6n^2}{n^6 - 1}$$

Proof: We have

$$\begin{aligned}
\frac{6n^2}{n^6 - 1} &= \frac{6n^2}{(n^3+1)(n^3-1)} \\
&= \frac{3n^2(n^3 + 1 - (n^3 - 1))}{(n^3+1)(n^3-1)} \\
&= \frac{3n^2}{n^3 - 1} - \frac{3n^2}{n^3 + 1}
\end{aligned}$$

□

Goal 0.54

Evaluate each of the following infinite products

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$$

Proof: we have $1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2} = \frac{(n-1)(n+1)}{n^2}$. □

Goal 0.55

Evaluate each of the following infinite products

$$\prod_{n=0}^{\infty} \left(1 + \frac{1}{2^{2^n}}\right)$$

Proof: As a hint, we compute $\left(1 - \frac{1}{2}\right) \prod_{n=0}^{\infty} \left(1 + \frac{1}{2^{2^n}}\right)$ instead. □

Goal 0.56

Evaluate each of the following infinite products

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$$

Proof: We factorize it and get

$$\frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$$

□

Goal 0.57

If $\sum a_n$ converges, then $\sum e^{a_n}$ diverges.

Proof: The statement is true because we know that $a_n \to 0$, and thus the right hand side is bounded below by sum of infinitely many 1's. □

Goal 0.58

If $\sum a_n$ converges, then $\sum a_n^2$ converges.

Proof: Construct $\frac{(-1)^{n-1}}{\sqrt{n}}$ which is a counterexample. □

Goal 0.59

If $\sum a_n$ converges and $\sum |b_n|$ converges, then $\sum a_n b_n$ converges.

Proof: The statement is true. □

Goal 0.60

If $f(x)$ is positive and continuous and $\displaystyle\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} f(x)$ converges.

Proof: The statement is false. □

Goal 0.61

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ then ($\sum a_n$ converges $\iff \sum b_n$ converges).

Proof: Consider the sequence that $a_{2n} = \frac{1}{\sqrt{2n}}$ and $a_{2n+1} = -a_{2n}$, and $b_n = \frac{1}{2n} + \frac{1}{\sqrt{2n}}$, thus we have constructed a counterexample. $\square$

Goal 0.62

If $\sum a_n$ converges then $\sum \frac{a_n}{1+a_n}$ converges.

Proof: False. $\square$

Theorem 0.19

Riemann Series Theorem

Example 0.39

We have

$$\int \frac{dx}{x(\ln x)^3} = \int \frac{d\ln x}{(\ln x)^3}$$

Example 0.40

Compute

$$\begin{aligned} \int \sin\sqrt{x}\, dx &= \int \sin\sqrt{x}\, d\left(\sqrt{x}\right)^2 \\ &= \int 2\sqrt{x}\sin\sqrt{x}\, d\sqrt{x} \end{aligned}$$

Example 0.41

Compute

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1+x}{\sqrt{1-x^2}}\, dx$$

Example 0.42

Compute

$$\begin{aligned}\int \cos^{2n}\theta \, d\theta &= \int \left(\cos^2\theta\right)^n \, d\theta \\ &= \int \left(\frac{1+\cos 2\theta}{2}\right)^n \, d\theta\end{aligned}$$

Example 0.43

Compute

$$\int (x \ln x)^2 \, dx = \int (\ln x)^2 \, d\frac{x^3}{3}$$

Lec 34 - Wed - Apr 3 - 2024

Example 0.44

Compute

$$\int (x \ln x)^2 \, dx = \int (\ln x)^2 \, d\frac{x^3}{3}$$

Lec 35 - Friday - Apr 5 - 2024

Example 0.45

Compute

$$\int_{\sqrt{3}}^{\sqrt{8}} \frac{\sqrt{1+x^2}}{x} \, dx$$

Take $x = \tan\theta$.

Example 0.46

Compute

$$\int_3^5 \frac{\sqrt{x^2-9}}{x^2} \, dx$$

Take $x = 3\sec\theta$.

Example 0.47

Compute

$$\int_{-\pi/6}^{\pi/6} \sin 2x \sin 3x \; dx$$

We have

$$\sin A \sin B = \frac{-\cos(A+B) + \cos(A-B)}{2}$$

Example 0.48

Compute

$$\int_0^{\pi^2} \sin^2 \sqrt{x} \; dx = \int_0^{\pi^2} \sin^2 \sqrt{x} \; d(\sqrt{x})^2$$

Example 0.49

Compute

$$\begin{aligned}\int_0^{\pi/2} \cos^{2n}(x) \; dx &= \int_0^{\pi/2} \cos^{2n-1}(x) \; d\sin(x) \\ &= \cos^{2n-1}(x)\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2} \sin(x)(2n-1)\cos^{2n-2}(x)(-\sin(x)) \; dx\end{aligned}$$

Example 0.50

Compute

$$\begin{aligned}\int_0^{\pi} \frac{\sin(nx)}{\sin(x)} \; dx &= \int_0^{\pi} \frac{\sin[(n-1)x + x]}{\sin(x)} \; dx \\ &= \int_0^{\pi} \frac{\sin(n-1)x \cos x}{\sin x} \; dx + \int_0^{\pi} \cos(n-1)x \; dx\end{aligned}$$

the expression on the left is symmetric about the line $x = \pi/2$.

Example 0.51

Compute

$$\int \frac{\sin x}{\sin x + \cos x} \; dx$$

We use Weierstrass Substitution, taking $t = \tan x/2$.

Example 0.52

Compute

$$\int_{\pi/4}^{\pi/2} \frac{x\ dx}{(\sin x + \cos x)^2}$$

Example 0.53

Compute

$$\int_1^2 (1+2x^2)e^{x^2}\ dx$$

Integration by parts.

Example 0.54

Compute

$$\int_1^3 \frac{\sqrt[3]{5-x}}{\sqrt[3]{5-x}+\sqrt[3]{1+x}}\ dx$$

we have

$$a+b = \left(a^{1/3}+b^{1/3}\right)\left(\left(a^{1/3}\right)^2 + \left(b^{1/3}\right)^2 - a^{1/3}b^{1/3}\right)$$